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Mathematical Tables
and other
Aids to Computation

A Quarterly Journal edited on behalf of the
Committee on Mathematical Tables
and Other Aids to Computation
by

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II • Number 15 • July 1946

Published by

THE NATIONAL RESEARCH COUNCIL

NATIONAL RESEARCH COUNCIL

DIVISION OF PHYSICAL SCIENCES

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Published quarterly in January, April, July and October by the National Research Council, Prince and Lemon Sts., Lancaster, Pa., and Washington, D. C.

All contributions intended for publication in *Mathematical Tables and Other Aids to Computation*, and all Books for review, should be addressed to Professor R. C. ARCHIBALD, Brown University, Providence, R. I.

Entered as second-class matter July 29, 1943, at the post office at Lancaster, Pennsylvania, under the Act of August 24, 1912.

The Electronic Numerical Integrator and Computer (ENIAC)

1. Introduction. It is our purpose in the succeeding pages to give a brief description of the ENIAC and an indication of the kinds of problems for which it can be used. This general purpose electronic computing machine was recently made public by the Army Ordnance Department for which it was developed by the Moore School of Electrical Engineering. The machine was developed primarily for the purpose of calculating firing tables for the armed forces. Its design is, however, sufficiently general to permit the solution of a large class of numerical problems which could hardly be attempted by more conventional computing tools.

In order easily to obtain sufficient accuracy for scientific computations, the ENIAC was designed as a digital device. The equipment normally handles signed 10-digit numbers expressed in the decimal system. It is, however, so constructed that operations with as many as 20 digits are possible.

The machine is automatically sequenced in the sense that all instructions needed to carry out a computation are given to it before the computation commences. It will be seen below how these instructions are given to the machine.

2. Description of the Machine as a Whole. The machine is a large U-shaped assemblage of 40 panels (see Figs. 1 and 2) which together contain approximately 18,000 vacuum tubes and 1500 relays. These panels are grouped to form 30 units (see Fig. 2), each of which performs one or more of the functions requisite to an automatic computing machine.

The units concerned mainly with arithmetic operations are 20 accumulators (for addition and subtraction), a multiplier, and a combination divider and square rooter.

Numbers are introduced into the ENIAC by means of a unit called the constant transmitter which operates in conjunction with an IBM card reader. The reader scans standard punched cards (which hold up to 80 digits and 16 signs) and causes data from them to be stored in relays located in the constant transmitter. The constant transmitter makes these numbers available in the form of electrical signals as they are required. Similarly, results computed in the ENIAC may be punched on cards by the ENIAC's printer unit operating in conjunction with an IBM card punch. Tables can be automatically printed from the cards by means of an IBM tabulator.

The numerical memory requirements of the machine are met in several ways. Three function table units provide memory for tabular data. Each function table has associated with it a portable function matrix with switches on which can be set 12 digits and 2 signs for each of 104 values of an independent variable. While primarily designed for the storage of tabular functions, the function table memory can be utilized for any numbers known before a computation begins. Switches for 20 digits and 4 signs on the constant transmitter can also be used for such numbers. Numbers formed in

the course of a computation and needed in subsequent parts of a computation can be stored in accumulators. Should the quantity of numbers formed during a computation and needed at a later time exceed the accumulator storage capacity, these numbers can be punched on cards and, later, can be reintroduced by means of the card reader and constant transmitter.

The ENIAC's programming memory (memory for instructions relevant to a particular computation) is chiefly local in nature, i.e. instructions are given to each unit. Sets of switches called program switches are located on the front faces of the various units. Before a computation begins these switches are set to specify which particular operations in a unit's repertoire are to be performed. Discussion of the manner in which the temporal order of operations (sequencing) is established will be postponed until section 7. A unit called the master programmer unit provides a certain amount of centralized programming memory. This topic, too, will be discussed at greater length in section 7.

The remaining two units, the initiating and cycling units, govern the operation of the others. The initiating unit has controls for turning the power on and off, initiating a computation, clearing the machine, and incorporates certain testing features. The cycling unit supplies the fundamental signals upon which the various units operate, and synchronizes the machine.

3. Synchronization. Every 1/5000th of a second the cycling unit emits a fundamental pattern of signals, nine trains of special pulses and a gate. The terms *pulse* and *gate* both refer to a change in voltage, either positive or negative, from some reference level. The distinction between these is their duration. In the ENIAC, pulses last for about 2 micro-seconds; gates, for 10 or more micro-seconds. The individual pulses of the various trains are spaced at 10 micro-second intervals.

One of the pulses emitted by the cycling unit, the central programming pulse (CPP), is of particular importance in this discussion. This pulse, emitted once every 1/5000th of a second, marks the beginning and end of a cycle. When a unit completes an operation, it transmits one of these as a program output pulse. Such a pulse, delivered to another unit, stimulates it to perform a given operation. The other signals supplied by the cycling unit are used in effecting this operation. For example, one of the trains of cycling unit pulses (the 9P) consists of 9 individual pulses, another train consists of only 1 pulse (the 1′ P). When an accumulator is called upon to transmit the number it stores, the accumulator does so by emitting appropriate numbers of the 9P and the 1′ P which originate in the cycling unit.

All the units have been so designed that they require an integral number of cycles to complete their operations. An accumulator, for example, requires 1/5000th of a second for an addition or subtraction. For this reason, 1 cycle or 1/5000th of a second is referred to as an *addition time*. The operation times for other numerical units of the ENIAC are summarized in the table below.

A number of separate trunk lines have been provided for the transmission of signals representing numbers (see section 6). For this reason and because the units operate in synchronism with one another, it is possible for a number of units to operate simultaneously. This, combined with the fact

OPERATION TIMES FOR ENIAC UNITS

Unit	Operation	No. of Addition Times 1 add. time = $\frac{1}{5000}$ sec.
Accumulator	Receives a number and adds it to number stored in accumulator or transmits the number or the negative of the number stored in it r times in succession where $1 \leqslant r \leqslant 9$.	r
Multiplier	Finds the signed product of a multiplicand with as many as 10 digits by a multiplier of p digits (where $2 \leqslant p \leqslant 10$).	$p+4$
Divider and Square Rooter	Finds p digits of the quotient or of twice the square root with the sign of the former.	Approximately $13[p+1]$ (depends on the digits of the arguments).
Function Table	Selects the functional value corresponding to a given 2-digit argument or any one of 4 neighboring values, and transmits it, or its negative, r times in succession.	$r+4$
Constant Transmitter	Emits a signed 5- or 10-digit number, previously read from a punched card or stored on its switches.	1

that the ENIAC depends for its operation not on the slow movement of mechanical parts, but instead on electronic circuits which operate on pulses spaced at 10 micro-second intervals, makes possible the high computing speed of this machine.

Although they do not take an absolutely definite number of addition times to complete operations, the electromechanical devices which operate with the ENIAC, namely the IBM card reader and card punch, have also been integrated into the synchronized system by providing for the emission of a programming pulse when one of these devices finishes reading or punching. It might be remarked that these devices operate at much lower speed than the ENIAC. Card reading requires approximately 0.5 seconds; punching about 0.6 seconds. To take advantage of the ENIAC's potentialities, it is desirable for computations or for card punching to proceed in parallel with card reading. There is no danger of initiating another set of operations until both the reading and its parallel operations have been completed since the reader does not emit a program output pulse until it has interlocked with the rest of the ENIAC by receiving a program pulse indicating completion of the parallel operations.

4. Electronic Elements and their Influence on Logical Characteristics. The circuits of the ENIAC are designed around a relatively small number of basic electronic elements. In the succeeding paragraphs we shall remark on a few of these elements which importantly influence the logical characteristics of the computer.

The simplest tube used is the triode, so called because it has three characteristic elements: the cathode (surface which gives off electrons), the plate (surface which receives electrons), and the grid (which controls the current passing through the tube). In addition, there is a heater to bring the cathode to the temperature required for it to emit electrons. Other tubes used in the ENIAC are multi-grid tubes, for example, the pentode which has 3 grids.

In all cases, vacuum tubes in the ENIAC circuits are used only as "on-off" devices instead of as amplitude sensitive devices, i.e. the presence or absence of a signal depends on whether a tube is conducting or not conducting, and not on any measured magnitude of current or voltage. To say that a tube is "on" or conducting means that with the usual convention of sign, current is flowing from the plate to the cathode. This implies that the plate is at a slightly higher voltage than the cathode, but that this voltage drop is small compared to the drop when the tube is turned "off" or non-conducting. Thus, if a tube is turned off, the voltage of the plate is raised and that of the cathode lowered. Hence, when a tube is turned off, the plate emits a positive signal, and the cathode one that is negative. If the tube is turned on, these signs are reversed. Within appropriate limits, a tube is conducting if its grid (or grids) is (or are) kept above a certain voltage, non-conducting if below that voltage. Thus, a tube is turned on by applying a positive signal to its grid (or grids), turned off by a negative signal.

The basic electronic memory device of the ENIAC is the *flip-flop* or trigger. A flip-flop consists essentially of a pair of triodes so connected that at any given time one, and only one, of the pair can be conducting. When a certain one of the tubes is conducting, the flip-flop is said to be in the normal state; when the other tube is conducting, the flip-flop is in the abnormal state. A flip-flop has two inputs and two outputs. A pulse received on one input (the set input) throws the flip-flop into the abnormal state, in which state it remains until restored to the normal state by a pulse received at its second (or reset) input. When the flip-flop is in the normal state, one output is positive and the other negative. In the abnormal state, the polarity of the outputs is reversed.

Corresponding to each flip-flop in the ENIAC, there is a neon lamp visible on the front of the machine. The neon is so connected to its corresponding flip-flop that it is lit when the flip-flop is in the *abnormal* state. These neons provide one of the most important visual checks on the operation of the computer since, as will be seen below, they provide an almost complete picture of the numbers in the machine and the operations being performed. In the normal method of operation, when the cycling unit emits its pulses at the 100 kilocycle rate, the neons blink on and off too rapidly for observation. However, a switch on the cycling unit enables the operator to switch the ENIAC into "one addition time" operation in which the fundamental signals for one addition time are emitted on demand, or into "one pulse time" operation in which the signals for an addition time are emitted in sequence, one each time a certain button is pushed. Either of these non-continuous methods of operation permit the operator, without disturbing the flip-flop memory, to stop the ENIAC at some point and examine the neons.

The *counters* of the ENIAC, in general, consist of a number of flip-flops arranged in sequence and so interconnected that the following characteristics result: (1) at any given time, one, and only one, flip-flop can be in the abnormal state: (2) the reception of a pulse at the input to the counter causes the flip-flop which is in the abnormal state to be reset and causes its successor to be set: and (3) the counter can be cleared so that a specific flip-flop comes up in the abnormal state and all others in the normal state.

Each flip-flop of a counter is called a stage and the reception of a pulse at a counter is said to advance the counter to the next stage. All counters in the ENIAC are ring counters, i.e., the first and last stages are so connected that if the counter is in its last stage and a pulse is received, the last stage is reset and the first stage is flipped into the abnormal state.

In accumulators, a 10-stage (decade) ring counter is used for each place of a 10-place number. Each stage of a decade counter corresponds to one of the digits between 0 and 9 inclusive. The sign of a number is handled by means of a PM counter which differs somewhat from the other ENIAC counters. The PM counter has 2 tubes, one for sign P (for positive numbers) and one for sign M (for negative numbers). Here each tube is referred to as a stage. The tubes are so connected that one, and only one, of them can be conducting at a given time. While the PM counter uses two tubes as does a flip-flop, it differs from an ordinary flip-flop in that it has but one input. The PM counter is also a ring counter. The 11 counters of an accumulator are connected by carry-over circuits so that when a counter advances from 9 back to 0, provision is made for the next counter at the left also to advance one stage.

Numbers are transmitted in the ENIAC in pulse form with the pulses for all ten decimal places and sign being given out at the same time. Trunk lines consisting of 11 conductors are used for the transmission of these pulses from one unit to another. To represent the digit $d(0 \leqslant d \leqslant 9)$ in a given place of a number, d pulses are transmitted over the conductor corresponding to that place. No sign pulses are used for positive numbers; nine sign pulses, for negative numbers.

An accumulator adds a number to one it already contains by counting the pulses it receives. Since the counters in an accumulator cycle in but one direction, subtraction is treated as a form of addition through the use of a system of *complements*.

Let us focus attention on the 10 decade counters of an accumulator. Any integer between 1 and $10^{10} - 1$ inclusive, is uniquely representable on the counters, but the numbers 0 and 10^{10} are indistinguishable. Hence the sum $(10^{10} - N) + N$ appears as 0 on the decade counters. The number $10^{10} - N$ is referred to as the complement of N with respect to 10^{10}.

The PM counter to the left of the decade counters permits a unique representation both of positive numbers N (for $0 \leqslant N \leqslant 10^{10} - 1$) and their complements. The PM counter can receive as its input either sign pulses (none for a positive number; nine for a negative number) or a carry-over pulse from the extreme left-hand decade. The reception of an even number of pulses leaves this counter in its original state; the reception of an odd number of pulses cycles the counter to its other stage. It is now easy to verify that all the usual arithmetic properties of addition and subtraction are obeyed in the system described. There follow a few illustrative examples: The sum of 801 and 527 appears as

```
P  0 000 000 801
P  0 000 000 527
-----------------
P  0 000 001 328.
```

The difference 801 minus 527 as

P 0 000 000 801
M 9 999 999 473

P 0 000 000 274;

and the difference 527 minus 801 as

P 0 000 000 527
M 9 999 999 199

M 9 999 999 726.

Counters are used not only for arithmetic purposes, but also as a part of the programming circuits which determine when and how a given unit shall perform. Each unit whose operations consume more than one addition time has such a program counter. While the unit is operating, a CPP is allowed to reach the input to the counter and cycle it one stage per addition time. Each stage of a counter, it is to be remembered, is a flip-flop. Therefore, when a counter reaches a given stage, the polarity of the outputs of the flip-flop for that stage is reversed from normal. Use is made of this to cause circuits to function in a manner appropriate to the addition time of the operation. When the program ring has been cycled through its range, it causes the unit to cease operating.

In addition to their use in program counters, flip-flops have a role in another aspect of the ENIAC's programming memory. However, before continuing with this topic, we should like to introduce a distinction between certain kinds of circuits in the various units. It is convenient in discussing the ENIAC to distinguish between the numerical circuits (which operate on signals representing numbers) and the programming circuits (which recognize when and how a unit is to operate and which then stimulate the numerical circuits to operate). In the case of most units, it is desirable to subdivide the programming circuit classification into program controls and common programming circuits. All but a few of the ENIAC units have multiple *program controls*, any one of which is capable of recognizing that a unit is to operate in a given way. All the program controls on a unit operate a common set of programming circuits which stimulate the numerical circuits to function in the desired manner.

Each program control consists of a set of program switches, a flip-flop, an input terminal for the reception of a program pulse, and associated tube circuits. Most program controls also have an output terminal for the transmission of a program output pulse when an operation is completed. These controls are referred to as transceiver program controls; the others, as receiver program controls. The outputs of the flip-flop in a program control are taken, ultimately, to the program switches. The points of these switches, in turn, are connected to various elements of the common programming circuits. When the flip-flops of all the program controls on a unit are in the normal state, the polarity of the signals delivered to the common programming circuits is such as to leave them quiescent.

When, however, a program control is stimulated by the reception of a program input pulse, the polarity of its flip-flop outputs is reversed, and, these signals, routed through the program switches, activate appropriate ones of the common programming circuits. Receiver program controls are used only in conjunction with operations that take but one addition time. One of the outputs of a receiver program control allows a CPP to reach the reset input and restore the flip-flop to its normal state one addition time after it is set. Transceiver controls operate in conjunction with a program counter. A signal from such a counter allows a CPP to reset the control after the required number of addition times have elapsed. When a transceiver program control is reset, it causes a CPP to be emitted as a program output pulse.

5. Numerical Units of the ENIAC. In this section we shall consider the units which perform numerical operations: addition, subtraction, multiplication, division and square rooting, and obtaining a number from the machine's memory. For each unit, we shall state the number of program controls in order to present some measure of the machine's capacity.

Each of the 20 accumulators is capable of storing a signed 10-digit number and, as described in section 4, of receiving a number and adding it to one it already contains. The accumulator has 5 input channels. For each input channel there is a digit input terminal consisting of 11 wires and a ground. The purpose of the multiple input channels will be treated in section 6. The accumulators also have two output circuits, with a digit output terminal for each one located on the front face. Through one output (the A output), the accumulator can transmit the number it stores; through the other (the S output), the complement of the number it stores. In any given addition time, the accumulator can either receive over one of its input channels or transmit through either or both of its outputs. The machine can be set up, as will be shown in section 6, so that any accumulator can communicate with any other, with the function tables, and with the constant transmitter. Communication to and from the multiplier and divider is possible only through specific accumulators.

In addition to the A and S outputs, the accumulators have a set of "static outputs" which provide a circuit directly from each stage of a counter. The static outputs of two specific accumulators which serve as argument accumulators are delivered to the multiplier. Static outputs for sign indication are delivered from another pair of accumulators to the divider and square rooter, and finally, the static outputs of a total of 80 decade counters and 16 PM counters are connected to the printer. Numbers which are to be printed are either formed in or sent to the accumulators connected to the printer before printing is due to take place. Any numbers stored in these accumulators are always available to the printer unit; the printer, however, takes cognizance of the numbers and causes them to be punched only when instructed to do so.

Each accumulator has 8 transceiver program controls and 4 receiver controls. These controls allow an accumulator to receive or transmit and to clear or not clear after transmission. A repeat switch included in the transceiver controls makes it possible to receive or transmit up to nine times in succession.

Each accumulator has a *significant figure* switch which enables the accumulator either to clear to 0 in all decades or to clear to 0 in all decades but one, selected by the switch setting, and to clear to 5 in that one. This, combined with the ability to delete the digit pulses in any desired place of a number (see section 6), makes it possible to round a number off in an accumulator. When it is desired, two accumulators can be interconnected in a special way to provide an accumulator with 20 decade counters and a PM counter and with 24 program controls.

The multiplier's numerical circuits consist essentially of those for storing products corresponding to one-digit arguments, for selecting the products appropriate to particular arguments and for emitting these products. This unit operates in conjunction with 4 accumulators (or possibly 6 if products with 10 to 20 digits are desired). Two accumulators, whose static outputs are delivered to the circuits for selecting products, hold the arguments. The multiplier unit produces the product of the entire multiplicand by one digit of the multiplier per addition time. Two multiplication tables are used, one for storing the tens digits of the fundamental products and one for storing the units digits of the products (for the product 4×7, for example, the tens digit is two and the units digit 8). The tens and units digits of the partial products are emitted simultaneously through two separate channels and accumulated separately in two accumulators. As multiplication by the digits of the multiplier proceeds, the multiplier shifts the partial products into the proper places before emitting them. When the digits of the multiplier are exhausted, the tens and units components of the products are added together in one of the 2 partial product accumulators.

The multiplier has 24 transceiver program controls. Each of these can be used to specify through which input channels argument accumulators shall receive their arguments for a given multiplication, whether the arguments are to be cleared after multiplication, how the product shall be disposed of from the final product accumulator, how many digits of the multiplier are to be used, and how many significant figures the product shall have.

The divider and square rooter operates in conjunction with 4 accumulators: one for storing the numerator (or radicand), one for the denominator (or twice-the-square-root), one for the quotient, and one for shifting (the necessity for this will be seen below). Division is performed by a method of successive subtractions or additions of the denominator to the numerator's remainder until an overdraft occurs. At that time the procedure is interrupted, the remainder from the numerator is sent to the shift accumulator for shifting one place to the left and then returned to the numerator accumulator. The divider emits components of the answer to the quotient accumulator, sending out $+1$ or -1 in a given place according as the denominator is subtracted from, or added to, the remainder. Square rooting is carried out by a method of successive subtractions or additions of odd numbers similar to that frequently used on desk model computing machines.

This unit has 8 transceiver program controls. Each control includes a program switch for specifying either division or square rooting and other program switches offering options similar to those offered by the multiplier program controls.

Both the multiplier and the divider and square rooter are semi-permanently connected to their associated accumulators for the communication

of numerical and programming signals involved in these operations. When these units are not in use, their associated accumulators are free to be employed in the same way as any others.

Each of the three function tables can store 104 entries consisting of 12 digits and 2 signs each. On its front face, the function table has a digit input terminal for receiving an argument and 2 digit output terminals. A sign, and 6 of the 12 digits, are emitted through each of these output terminals. By means of special adaptors (see section 6) the outputs of both terminals may be combined so that entries for one or more functions may be derived from each function table.

Each of 11 transceiver program controls on the function table permits the operator to specify transmission of the functional value corresponding to a particular value of the argument or to any of 4 neighboring values of the argument, or to specify transmission of the negative of the preceding. Multiplication by small constants is also possible along with the transmission since each control includes a repeat switch for specifying transmission up to 9 times in succession. The options afforded by the function table's program controls make it easy to set up the machine for interpolation routines.

The constant transmitter has 30 transceiver program controls. Twenty-four of these give access to the total of 16 signs and 80 digits which the reader reads from a punched card. These 24 controls are arranged in groups of 6 which operate common programming circuits for selecting any signed 5- or 10-digit group out of a total of 4 signs and 20 digits. The remaining 6 program controls are used for the 4 signs and 20 digits which can be set up on the constant transmitter's switches.

6. Interconnections for a Particular Computation. Before the ENIAC can carry out a computation, not only must the program switches be set, but also the various units of the ENIAC must be connected for the communication of numerical and programming signals in such a way as to meet the needs of the particular computation. Multiple trunk lines running past the units are provided for this purpose. Both the digit and program trunk lines consist of 11 conductors each, mounted on trays. The digit trunks have 12-point terminals spaced so that there is one in front of each panel. These terminals are connected to digit terminals on the units by digit cables. The program trunks have a group of 11 separate 2-point terminals instead of the 12-point digit terminals. Each of these terminals can be connected to a program pulse input or output terminal by means of a program cable. Within limits, any number of digit terminals can be plugged into the same digit trunk or any number of program terminals into the same line of a program trunk. The program trunks are stacked below the panels containing the program switches and the digit trunks above the switch panels (see Fig. 1).

Adaptors, consisting of specially wired plug and socket assemblies, are also used in the transmission of digits. Each conductor of a digit tray ordinarily carries the pulses for a specific place of a 10-place number or for sign indication. Adaptors are used when it is desired to effect a special relationship between the digit places of the transmitting and receiving unit. Shifter adaptors are used to effect multiplication by powers of 10. Deleter adaptors

open certain circuit lines so as to eliminate the pulses for one or more places of the number.

The five separate digit input channels on an accumulator make it possible for an accumulator to receive from any one of 5 different digit trunks. This makes it possible for numerical operations to be carried on in different parts of the machine simultaneously. The different input channels also are useful in that they enable reception through different adaptors at various times.

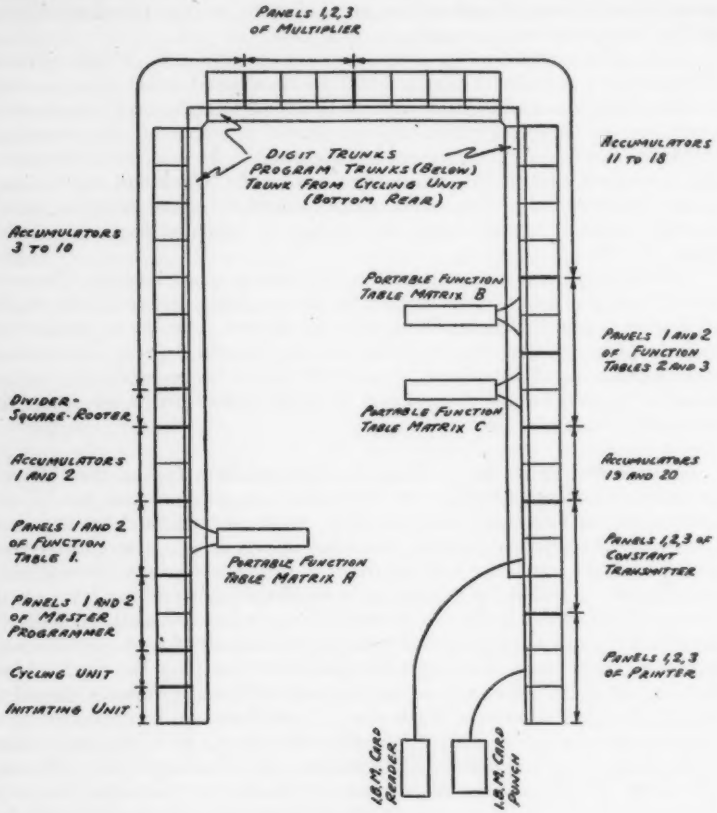

PANELS 1,2,3 OF MULTIPLIER
DIGIT TRUNKS
PROGRAM TRUNKS (BELOW)
TRUNK FROM CYCLING UNIT (BOTTOM REAR)
ACCUMULATORS 11 TO 18
ACCUMULATORS 3 TO 10
PORTABLE FUNCTION TABLE MATRIX B
PANELS 1 AND 2 OF FUNCTION TABLES 2 AND 3
DIVIDER-SQUARE-ROOTER
PORTABLE FUNCTION TABLE MATRIX C
ACCUMULATORS 1 AND 2
ACCUMULATORS 19 AND 20
PANELS 1 AND 2 OF FUNCTION TABLE 1.
PORTABLE FUNCTION TABLE MATRIX A
PANELS 1,2,3 OF CONSTANT TRANSMITTER
PANELS 1 AND 2 OF MASTER PROGRAMMER
PANELS 1,2,3 OF PRINTER
CYCLING UNIT
INITIATING UNIT
I.B.M. CARD READER
I.B.M. CARD PUNCH

FIG. 2.

7. Sequencing and the Master Programmer. We shall consider in this section the procedure for carrying out a computation by means of this machine. For the ENIAC, as for human computers, the computation must consist of a series of arithmetical operations. Thus, for example, if the computation involves evaluating integrals, numerical integration formulae must be introduced. Another preliminary step before the computation can

be given to the machine consists of planning for the storage of numbers required in the computation by means of the function tables, the accumulators, and possibly, punched cards. Incidental to this is the consideration of establishing digit connections between the various units so as to make possible the required transfers of numbers from one part of the machine to another.

We next consider how to instruct the ENIAC in its routine for a particular computation. One or more program controls must be devoted to each arithmetical operation of the computation. For example, if accumulator 1 stores the number a and accumulator 2 the number b, then, to form $a-b$, accumulator 2 must be instructed to transmit through its S output, and accumulator 1 must be instructed to receive over a channel connected to the S output of accumulator 2.

To tie the individual operations into a *sequence* in which one operation or a group of operations is initiated upon the completion of another, use is made of the transceiver program controls. Recall that when a transceiver control completes an operation, it emits a program output pulse. If the program output pulse terminal of such a control is connected to a line in a program trunk, all program controls whose input terminals are connected to this line are stimulated when the transceiver control completes its operation. To illustrate the method of sequencing the ENIAC, let us consider the trivial computation for simultaneously generating a table of squares and cubes. We assume that n, n^2, and n^3 are stored in certain accumulators and proceed inductively to form $n+1$, $(n+1)^2$, and $(n+1)^3$. The steps for these computations are shown schematically in Fig. 3.

The questions that remain to be considered for this computation are: (i) How is the computation initiated?; (ii) How could this sequence be repeated r times?; and (iii) How could a different sequence be initiated after the rth repetition? The first matter is taken care of by means of a program control on the initiating unit which, under the control of a push button, causes an initiating pulse to be emitted. By delivering the initiating pulse to program line A-1, we could provide for starting the computation. The problems of iterating the sequence into a *chain* and linking another sequence or chain to it are provided for by the master programmer.

The *master programmer* consists of ten units, each of which is capable of counting program pulses and of switching program connections. Each unit has a 6-stage counter, called a stepper counter and may have from 1 to 5 associated decade counters which are connected to one another for carry-over purposes. For each stepper-decade combination, there is one input terminal and there are 6 output terminals. When a pulse is received at the stepper, it is registered in the decades and a pulse is emitted through one of the 6 output terminals. The stage of the stepper counter determines through which output terminal the pulse is emitted.

Control of the stage of the stepper counter is dependent upon the following considerations: Associated with each stage of a stepper counter is a group of so-called decade switches. Let us say that the setting of the decade switches associated with stage s of a stepper counter is d_s $(0 \leqslant d_s < 10^5)$. Then, if the stepper counter is in stage s, it emits a pulse from the output terminal associated with stage s on each of d_s occasions. Input pulse number d_s, moreover, causes the stepper counter to advance to stage $s+1$ and causes

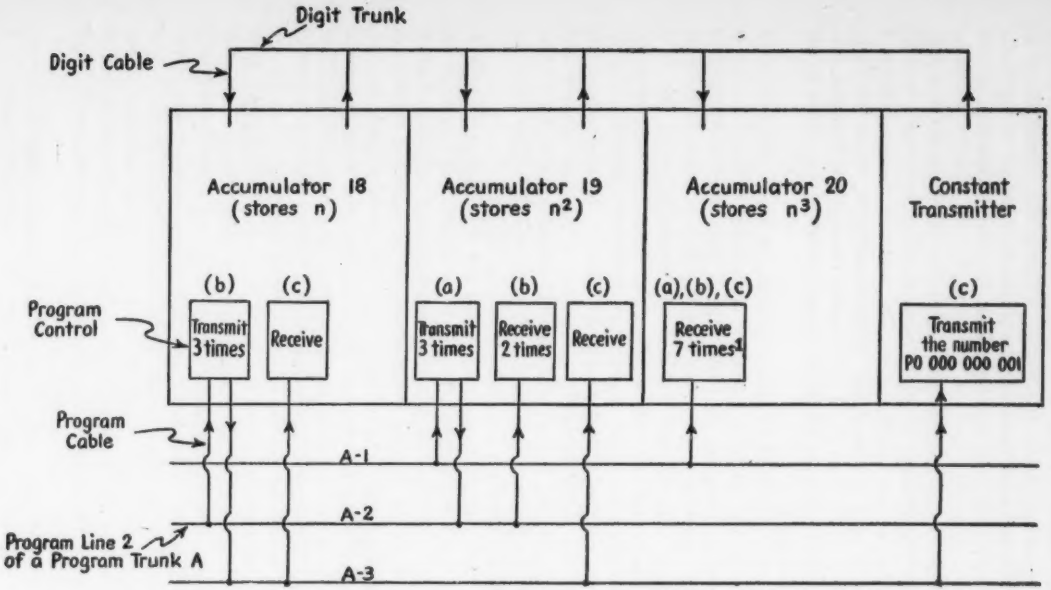
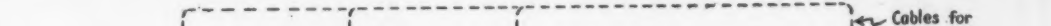


FIG. 3.

(*a*) Accumulator 19 transmits and accumulator 20 receives $3n^2$ so that accumulator 20 stores $n^3 + 3n^2$.
(*b*) Accumulator 18 transmits the number n three times. Accumulator 19 receives twice so that this accumulator stores, as a result, $n^2 + 2n$. Accumulator 20 receives $3n$ so that it now stores $n^3 + 3n^2 + 3n$.
(*c*) The constant transmitter transmits the number 1 and accumulators 18, 19, and 20 receive it. As a result they now hold $n + 1$, $(n + 1)^2$, and $(n + 1)^3$ respectively.

[1] A less economical procedure for achieving the same result would use 3 program controls: one set up to receive 3 times and stimulated from A-1, a second set up to receive 3 times and stimulated from A-2, and a third set up to receive once and stimulated from A-3.



and stimulated from A-1, a second set up to receive 3 times and stimulated from A-2, and a third set up to receive once and stimulated from A-3.

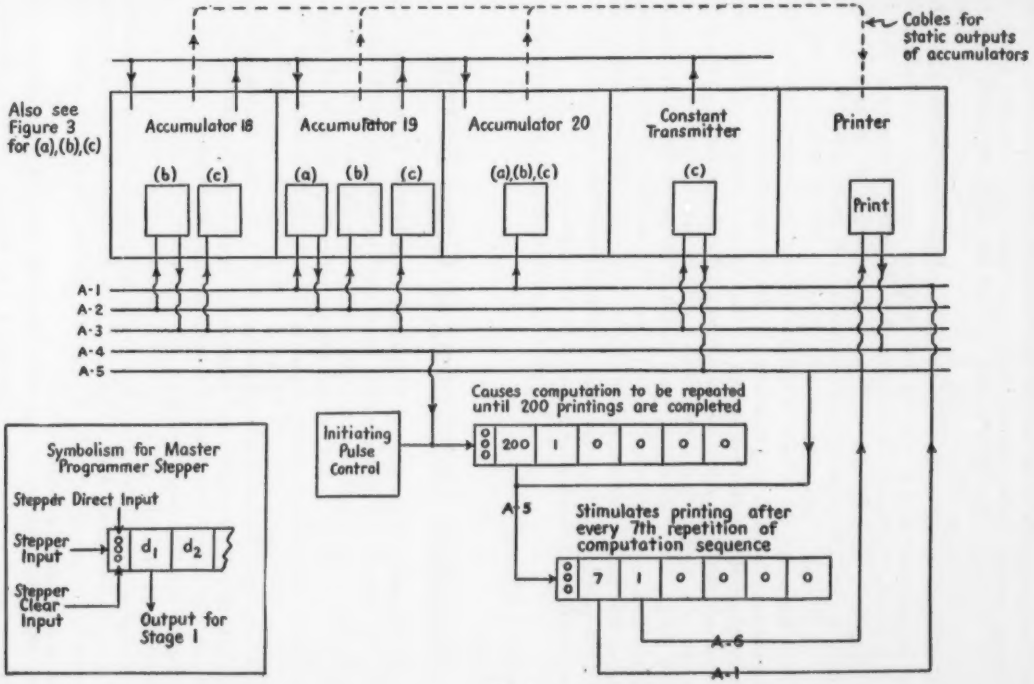


FIG. 4.

the associated decade counters to clear to 0. Besides the preceding, each stepper counter has a direct input which allows the stepper counter to be stepped independently of the decade counters and a clear input which allows the stepper counter to be cleared back to its first stage from any stage. It might be remarked that digit pulses, as well as program pulses, may be brought to the stepper direct input.

We will now return to the problem discussed above to illustrate how the master programmer may be used to link program sequences. Let us suppose that it is desired to repeat the computations described 7 times and then to print. Suppose, furthermore, that computations are to cease after 200 printings. A complete plan for this computation is given in Fig. 4.

In the ENIAC, programs may be linked together either serially, as described above, or the choice of routine may be made to depend on the magnitude of a number. The criterion may be the magnitude of the digit in a specific place of a number or it may be the sign of a number. Either form of magnitude discrimination can be accomplished by sending pulses from some lead of a digit output terminal to a stepper direct input.

The ENIAC has, of course, been designed to handle far more complicated and tedious computations than the one discussed above which required the use of less than 4 percent of the machine's programming capacity. This illustrative problem is, however, useful for indicating how the machine can be instructed in its routine for a particular computation.

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Scientific Computing in Great Britain

1. Introduction: Commercial Machines in Scientific Computing. The commercial calculating machine has been used in scientific work, after a fashion, for many years. Twenty years ago one or two hand-operated machines were to be found in the mathematics departments of British universities, used by graduate students. But a great stimulus to their better and wider use was given by L. J. COMRIE in lectures given at the London School of Economics in 1926[1] and 1927,[2] in which he stressed the importance of the intelligent and resourceful use of commercial machines.

About the same time Comrie described the "end-figure" method[3] of constructing mathematical tables by subtabulation, and a means of using this method on Hollerith punched-card equipment. Four years later[4] came his description of the Burroughs Class 11 machine and its use for integration from second differences. About the same time he described his application of the Hollerith tabulator to E. W. BROWN, *Tables of the Moon*, which still stands as a remarkable achievement.

His next great advance[5] was the use of the six-register National Accounting machine as a "difference-engine" to handle problems in finite differences. As a result there are now fifteen National machines in use in research establishments in Great Britain, several times as many as there are in similar use in the United States, where the machine originated!

The story of the adaptation of commercial machines to uses in the scientific field in Great Britain is almost wholly the story of Comrie's life

and work. The debt owed to him for this, even without considering the great work he has done in table-making and in discovering errors in published tables, is hard to estimate and impossible to repay.

2. Finite Differences and Computing Methods. In this, as in other sections, rather than attempt a comprehensive survey, two or three examples will be chosen, as being of unusual importance, or having some topical interest.

From Comrie's work we choose a device, known as the throw-back, for simplifying ordinary interpolation formulae involving differences.[3] Although Comrie invented this device independently, he later discovered that a suggestion for its use had, in fact, been made some years earlier by that great astronomer E. W. Brown. Its advantages, as given by Comrie, are:

> "It enables whole columns of differences to be dispensed with in published tables, and saves the user the trouble of looking out (in general) two coefficients and performing two multiplications."

The method can be applied either to Bessel's or to Everett's formula. Bessel's formula can be written in the following form, if, instead of n, the fraction of the tabular interval, we write $y = n - \frac{1}{2}$ and separate the terms involving odd and even differences:

$$u_n = u_0 + n\Delta_1' + \frac{y^2 - \frac{1}{4}}{2.2!}(\Delta_0'' + \Delta_1'') + \frac{(y^2 - \frac{1}{4})(y^2 - \frac{9}{4})}{2.4!}(\Delta_0^{iv} + \Delta_1^{iv}) + \frac{y(y^2 - \frac{1}{4})}{3!}\Delta_1''' + \frac{y(y^2 - \frac{1}{4})(y^2 - \frac{9}{4})}{5!}\Delta_1^{v} + \cdots.$$

If we denote the coefficients of the differences of second and higher orders by B'', B''', $\cdots$, etc., then

$$\frac{B^{(2n+2)}}{B^{(2n)}} = \kappa_{2n} = -\frac{\frac{1}{4}(2n+1)^2 - y^2}{(2n+1)(2n+2)},$$

$$\frac{B^{(2n+3)}}{B^{(2n+1)}} = \kappa_{2n+1} = -\frac{\frac{1}{4}(2n+1)^2 - y^2}{(2n+2)(2n+3)}.$$

Since the limits of variation of y are $\pm\frac{1}{2}$, these ratios vary only a little; they vary proportionately less as n increases. If they were strictly constant, fourth differences could be accounted for exactly by writing

$$B''(\Delta_0'' + \Delta_1'') + B^{iv}(\Delta_0^{iv} + \Delta_1^{iv})$$

as

$$B''[(\Delta_0'' + \Delta_1'') + \kappa_2(\Delta_0^{iv} + \Delta_1^{iv})] = B''(M_0'' + M_1'')$$

where

$$M'' = \Delta'' + \kappa_2\Delta^{iv}$$

and both fourth- and sixth-order differences could be accounted for by writing

$$M'' = \Delta'' + \kappa_2\Delta^{iv} + \kappa_3\kappa_4\Delta^{vi}.$$

If suitable mean values of κ_2 and κ_4 are chosen, the true modified differences are represented sufficiently well for the residual errors to be neglected. A full table, giving the upper limits of differences with which these modifications may be safely used, is given in the original paper.

The principal drawbacks of the method, as stated by Comrie, are that additional computation is necessary to produce the modified differences and that more responsibility is thrown on the proof-reader, as the modified differences cannot be so readily checked from the other columns of the table. Modified differences have been used extensively in the well-known B.A.A.S., *Mathematical Tables*. In 1932 AITKEN published a method of interpolation[6] which is most attractive in its simplicity. It is not a formula, but a set of rules to be followed, and, furthermore, the process is exactly the same for both direct and inverse interpolation. It consists in a succession of interpolations by proportional parts. If we have two values u_a and u_b and wish to find the value of u_x, we have

$$u_x = u_a + \frac{x-a}{b-a}(u_b - u_a)$$

$$= \frac{b-x}{b-a}u_a + \frac{x-a}{b-a}u_b$$

i.e. the weighted mean of u_a and u_b. This can be written as

$$u_x = \begin{vmatrix} u_a & a-x \\ u_b & b-x \end{vmatrix} \div (b-a);$$

i.e. a cross-product divided by the difference between the right-hand pair of factors. Aitken calls this the linear cross-mean. It is a very simple operation on an ordinary desk calculator.

We will denote this linear cross-mean by $u_x(a, b)$. Now if we have a series of values $u_a, u_b, u_c, \cdots$ etc. and take them in pairs we can form several cross-means, each of which will be an estimate of the true value of u_x. The calculation is set out as shown in the table below:

Argument	Function	Cross-Means	Parts
a	u_a		$a-x$
b	u_b	$u_x(a, b)$	$b-x$
c	u_c	$u_x(a, c)$	$c-x$
d	u_d	$u_x(a, d)$	$d-x$

We now treat the cross-means as though they were tabular values, and form cross-means from them, i.e. we calculate

$$u_x(a, b, c) = \begin{vmatrix} u_x(a, b) & b-x \\ u_x(a, c) & c-x \end{vmatrix} \div (c-b)$$

and $u_x(a, c, d)$ similarly.

Aitken has shown that this process is the same as using the general divided-difference formula, to the same order of differences as the number of times the process is performed. It can be continued indefinitely, until the linear cross-means have converged to the desired degree of approximation. A numerical example taken from Aitken's paper, is given below.

Adjusted Argument	u	Stage (1)	Stage (2)	Stage (3)	Stage (4)	Parts
−2	5772 1566					−2.68327
−1	5608 8546	53 339732.4				−1.68327
0	5447 8931	.. 371133.4	39 2588.8			−0.68327
1	5289 2109	.. 401987.0	.. 2128.1	227 4.0		0.31673
2	5132 7488	.. 432306.6	.. 1674.9	... 6.6	..3.2	1.31673
3	4978 4499	.. 462107.2	.. 1229.9	... 9.3	..3.2	2.31673

This method is worthy of serious attention now that the automatically controlled calculating machine is an accomplished fact. For these machines a method is needed which involves the minimum of tabular matter, and a simple set of instructions to the machine. Aitken's method fulfils both these requirements. It converges rapidly, and is self-checking. It is not a formula, but a set of instructions simple enough to be incorporated into a machine routine, requiring neither tables of coefficients nor tabulated differences. Since it does not depend upon the function being tabulated at equal intervals of the argument, the same machine routine can be used both for direct and inverse interpolation. The fact that it requires more multiplications than conventional difference formulae is no obstacle, in view of the speeds of machine operation now contemplated.

In describing the process, Aitken says, "The convergence of the interpolation is indicated to the eye of the computer by the convergence towards equality of consecutive cross-means in the same column at any stage; the process is stopped when such entries agree to an assigned number of digits." It is evident that the eye of the machine could observe this indication equally well, and that such a self-checking method is particularly well suited to the machines of the future. Aitken also says, ". . . in the later stages, when cross-means in the same column converge to agreement in their more important digits, we need form cross-means of the later and differing digits only, since the digits that are common to both will be reproduced under the operation of weighted mean and therefore need be recorded in the first entry only. . . ." This also is within the power of the machines now contemplated.

No better example could be found of the importance of suiting computing methods to the equipment being used. Just as great advantages can be derived from skilful exploitation of commercial equipment by devising methods appropriate to the facilities they provide, so must the most efficient methods be found for the automatically controlled machines now being developed.

3. Differential Equations. Some of the most interesting developments in numerical methods in Great Britain have been in the solution of differential equations. This may seem strange, in view of the fact that the differential analyser is an American development, but is nevertheless true. Between 1912 and 1928 RICHARDSON published several papers on a general method, applicable to both ordinary and partial differential equations, which he christened "The deferred approach to the limit."[7] The basic principle is both simple and powerful. If, in any ordinary or partial differential equation, the differential coefficients are replaced by finite differences of the same order, using steps of a certain size in the independent variable, then the solution obtained is not the solution of the problem described by the differential equation, but of a corresponding problem in finite differences. Let the solution we require be ϕ_0, and the solution of the corresponding problem in finite differences be ϕ_h where h is the size of the interval. Then, if the differences are properly centered, Richardson proves for some problems, and suggests that for many others,

$$\phi_h = \phi_0 + \phi_2 h^2 + \phi_4 h^4 + \cdots$$

a series in even powers of h, in which the ϕ_2, ϕ_4, etc., are independent of h.

It is practicable, therefore, instead of choosing such small intervals in h that ϕ_2 and all higher terms are negligible, to work with a comparatively coarse mesh, repeat the calculations for steps of a different size, and then extrapolate to steps of zero size; for instance, if we have two solutions for interval h, and $\frac{1}{2}h$ respectively, then

$$\phi_h = \phi_0 + h^2\phi_2 + \cdots,$$
$$\phi_{\frac{1}{2}h} = \phi_0 + \tfrac{1}{4}h^2\phi_2 \cdots,$$
$$\phi_0 = \phi_{\frac{1}{2}h} + \tfrac{1}{3}(\phi_{\frac{1}{2}h} - \phi_h).$$

Thus when one solution has steps half the size of the other, we have the simple rule: add one third of the difference to the better approximation.

An interesting and simple application of the method is to calculate π by taking the perimeters of the inscribed square and hexagon in a circle, assuming that the error is inversely proportional to the square of the number of sides. This gives 3·18, which is astonishingly accurate when we consider the crudeness of the two original approximations, namely 2·83 and 3·00. A defect of the method is its proneness to oscillatory errors in open-boundary problems, unless care is taken. In many problems it can be extremely powerful, but one cannot always find limits for the error.

HARTREE and WOMERSLEY[8] have combined this method with the use of the differential analyser for the solution of certain types of partial differential equations. If we take the ordinary diffusion equation as an example,

$$\frac{\partial\theta}{\partial t} = \frac{\partial^2\theta}{\partial x^2}$$

and replace the time derivative by a finite difference, we have

$$\frac{\theta_1 - \theta_0}{h} = \tfrac{1}{2}\left(\frac{\partial^2\theta_1}{\partial x^2} + \frac{\partial^2\theta_0}{\partial x^2}\right)$$

and thus, if θ_0 is known as a function of x for $t = t_0$, say, we have an ordinary differential equation for θ_1 at $t = t_1$. We thus divide up the domain over which the solution is required into a series of strips, and integrate the ordinary differential equations in succession, repeating the whole process for strips of a different size and then extrapolating to strips of zero width. This has been successful in the boundary layer problem, and in many diffusion problems. In general, it is best suited to problems of hyperbolic type, or open-boundary problems.

Another method of solving partial differential equations, applicable to closed boundary conditions, is SOUTHWELL's relaxation method.[9] Of it, TEMPLE said,[10] ". . . one of the most powerful methods of computation in mathematical physics and engineering." This technique began as a method of finding the displacements of complicated elastic structures, and has been extended by Southwell and his pupils to cover a remarkably wide field of problems. The basic idea is very simple. Although it may be very difficult to calculate the displacements corresponding to given forces, it is easy to calculate the forces which maintain given displacements. These "constraints" on the elastic system are then "relaxed." The displacement of the system is altered so that the largest of the "unwanted" forces is

eliminated. Attention is then directed to the largest force remaining, and this is "relaxed" in turn. The process is continued until the remaining "unwanted" constraints lie within the limits of error of the original data. In his book *Relaxation Methods in Engineering Science*, Southwell illustrates by carefully-chosen examples how, by relaxing the constraints in groups, the "relaxer" can make use of his knowledge of the physical nature of the problem to reduce the time required to reach the final solution. It would be quite impossible to name the many types of problem to which these methods have been applied, but they include surveying errors, natural frequencies of oscillating systems, and two-dimensional solutions of Laplace's and Poisson's equations, including plastic torsion. Conformal transformations have been treated by it, biharmonic analysis including stress-distributions in solids of revolution, and the flow of a compressible fluid through a two-dimensional nozzle. Southwell has gone so far as to call this method "a new mathematics for engineering science." As it is a method of successive approximation, it is self-checking, and it would, perhaps, be possible to make it fully automatic but for the fact that to get the best results from it considerable judgment and experience are required.

In the most recent paper[11] of the long series describing applications of these methods, it is pointed out that no paper in the series gives an exact account of relaxation methods as now applied to problems in two independent variables, and a summary is given of current methods. The basic principle is still the same. An approximation in finite differences is substituted for the partial differential equation. In effect, a system of equations is obtained which links together values of the wanted function at nodal points of a regular lattice or net. These equations are then solved by the relaxation technique. In this paper, meshes of varying size are used systematically, and a definite technique established for using specially fine meshes in the parts of the region where they are particularly needed.

If this method is to be made fully automatic, it will be necessary to construct a machine which can make a choice of one of several possible calculations according to the results of work done previously. Not only this, but the machine would have to have a vastly greater storage capacity than any present equipment. It would need to have storage mechanism, readily available, for at least a thousand numbers. It seems doubtful whether, with present technique, such a machine would be practicable, in view of the large quantity of equipment that would have to be kept continuously in service.

4. Special Machines. Although the differential analyzer was developed in the United States, there is one development in England that is worth mentioning. This is the small, cheap, differential analyzer, surprisingly accurate in relation to its first cost, to be seen in two of our universities. The first of these was made by HARTREE and PORTER in Manchester, and aroused a great deal of interest. Its accuracy was in the region of 2 to 5 per cent and it was constructed almost entirely of Meccano—a constructional toy. Another similar but more carefully designed machine was built in Cambridge, and turned out some excellent work. Both these machines were intended as models on which to gain experience during the period in which the large machines for these universities were being constructed, but

they have continued to exist in their own right. H. W. S. MASSEY built a third machine at Queen's University, Belfast. This was brought by him to University College, London, just before World War II and was unfortunately destroyed by enemy action.

The calculation of serial correlation coefficients has become an important practical problem in recent years. In 1936 Foster[12] described a simple integrating mechanism for integrating the square of a function which he showed could be adapted for this purpose. It uses the wheel and disc integrator, now familiar on the differential analyzer. Imagine an ordinary planimeter wheel running on a 2-inch (paper covered) disc, and sliding on a rod on which is a pointer for setting on the curve. The curve is moved under the pointer in small finite steps (about $\frac{1}{10}$-inch). At each step the pointer is set on the curve, and the depression of a key traverses the curve and rotates the disc by a small amount. The planimeter wheel therefore integrates the function and gives its mean value. To obtain the mean square, another disc and wheel are used, resting on the first wheel. These are so arranged that the displacement of the upper wheel from the centre of the upper disc is the same as the displacement of the lower wheel from the lower disc. The upper disc is driven by the lower wheel, and therefore is rotated an amount proportional to the displacement of the lower wheel. Thus the amount by which the upper planimeter wheel is rotated at any step is proportional both to its displacement and the amount of rotation of the upper disc, i.e. to the square of the displacement of the lower wheel. To calculate serial correlation coefficients, three of these integrators are used, linked together by an averaging motion. If the displacements of the two outer integrators are x and y respectively, then the displacement of the third is $\frac{1}{2}(x + y)$, and thus one run of the curve through the device gives simultaneously Σx^2, Σy^2 and $\frac{1}{2}\Sigma(x + y)^2$ from which all the quantities for calculating the correlation coefficient are obtained.

Two other instruments make use of optical methods for constructing periodograms. The first of these, described by Foster[13] in 1930, is an instrument for periodogram analysis. The observations to be analyzed are made into a long narrow "lantern slide," rather like a "variable area" sound track, somewhat extended and magnified. This is illuminated by diffused light, which then falls upon a harmonic grating, i.e. a grating in which the light transmitted is proportional to $(1 + a \cos 2\pi ux)$. (Such gratings are prepared by photographing a sine curve with a short-focus cylindrical lens in front of the camera lens.) The light then falls upon a ground-glass screen. The grating can be moved along an optical bench so that its position relative to the screen and the source can be varied. It can be shown that the illumination on the ground-glass screen will be in form of a series of light and dark fringes if there is a period in the original data which bears the same ratio to the period of the grating as the ratio of the distances of the source and the screen from the grating. The stronger the period, the more intense the bands on the screen. By this method, outstanding periodicities in textile yarns, due to machine faults, can be traced rapidly. If, instead of a harmonic grating, a grating could be used which gives a transmission proportional to the height of the original curve, suitably contracted (say in the ratio 2 : 3) along its length, then the grating, if placed at the corresponding distance, will display a series of light and dark bands, the illumi-

nation at any point being proportional to the corresponding serial correlation coefficient.

MARTINDALE[14] has described a correlation periodograph which makes use of this principle. The grating is replaced by a replica of the curve on a reduced scale and a cylindrical lens. The amount of light is measured by replacing the screen by a narrow vertical slit, long enough to collect all the light passing through the highest ordinates of the curves, and collecting the light on to a photo-cell. The photoelectric current is amplified and measured on a galvanometer. The measurements give accurate relative values of the correlation coefficients; but the present arrangement of the instrument does not permit of calibration to give absolute values.

Descriptions of the three machines have been collected in a paper read before the Research Section of the Royal Statistical Society.[15] At the same meeting CUNNINGHAM and HYND[16] gave a description of a relay machine, devised by BARNES and WEIR, for calculating serial correlation coefficients. In this the observations (which must lie within the limits ± 63) are coded on five-hole teletype tape in binary notation, and multiplication is carried out in the binary scale. The results are displayed on indicating lamps. The speed is one multiplication per second, but even this is much slower than the speed that can be achieved on standard punched-card equipment with less limitation on the observations.

5. Conclusion. There is a great need, with the recent growth of the use of numerical methods, for a critical and exhaustive survey of computing methods, with a study of their field of usefulness in connection with present-day equipment. The examples given are only "small samples" from a vast mass of published material. Before undertaking the design of special machines performing single calculations, the methods and equipment already available should be carefully studied. It is also worth stressing that the great new machines now under development should be provided with "instructions" based on methods suited to their capabilities. Aitken's method of interpolation is a particularly good example of the kind of method that should be looked for, and in the future it will be worth while to think in terms of processes of this kind, rather than formulae in the conventional sense.

J. R. WOMERSLEY

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More Zeros of Certain Bessel Functions of Fractional Order

The following table is a continuation of the one given by M. ABRAMOWITZ in *MTAC*, v. 1, 1945, p. 353–354. The quantities $j_{\nu,s}$, where $j_{\nu,s}$ denotes the sth zero of $J_\nu(x)$, are given below for $\nu = \pm \frac{1}{4}, \pm \frac{1}{3}, \pm \frac{2}{3}, \pm \frac{3}{4}$, and for $s = [8(1)30; 10D]$. These values are believed to be correct to well within a unit in the tenth decimal. Thus this table supplements the one given by Abramowitz up to $s = 7$ or 8. The roots $j_{\nu,s}$ were calculated from the well-known asymptotic expansion (see *MTAC* p. 354), carrying 12D in the work.

An interesting check was performed upon these roots by obtaining the first difference as a function of s and then obtaining the divided differences of those first differences as functions of ν. Apart from the first entry corresponding to $s = 8$ and which was checked independently, the divided differences (which were, in turn, also differenced as a function of s where they were largest, i.e. for s near 8), showed an error very much within 2000 units in the twelfth decimal place. Since the error in the roots was multiplied by a probable factor of 200 or more as a result of the process of taking the divided difference, the roots in the original manuscript are probably correct to about a unit in the 11th decimal place.

Most of the calculations on these roots were performed by Miss RUTH ZUCKER and Mrs. RUTH CAPUANO.

Tables of $j_{\nu,s}$

s	$\nu = \frac{1}{4}$	$\nu = -\frac{1}{4}$	$\nu = \frac{3}{4}$	$\nu = -\frac{3}{4}$
8	24.7438277961	23.9585534953	24.8737314228	23.8266562471
9	27.8849946034	27.0996936674	28.0150117149	26.9679102166
10	31.0262474761	30.2409276522	31.1563548787	30.1092347166
11	34.1675627296	33.3822290291	34.2977436760	33.2506098032
12	37.3089246363	36.5235804430	37.4391666407	36.3920224013
13	40.4503223437	39.6649700250	40.5806158496	39.5334635829
14	43.5917481224	42.8063893836	43.7220856531	42.6749270440
15	46.7331963179	45.9478324186	46.8635719141	45.8164082072
16	49.8746626986	49.0892945893	50.0050715339	48.9579036683
17	53.0161440346	52.2307724460	53.1465821455	52.0994108430
18	56.1576378181	55.3722633210	56.2881019100	55.2409277345
19	59.2991420721	58.5137651192	59.4296293768	58.3824527754
20	62.4406552173	61.6552761718	62.5711633865	61.5239847181
21	65.5821759767	64.7967951336	65.7127030012	64.6655225573
22	68.7237033066	67.9383209080	68.8542474541	67.8070654737
23	71.8652363457	71.0798525921	71.9957961121	70.9486127933
24	75.0067743771	74.2213894359	75.1373484480	74.0901639563
25	78.1483167988	77.3629308110	78.2789040192	77.2317184937
26	81.2898631020	80.5044761872	81.4204624512	80.3732760100
27	84.4314128537	83.6460251137	84.5620234254	83.5148361692
28	87.5729656827	86.7875772052	87.7035866686	86.6563986839
29	90.7145212696	89.9291321301	90.8451519455	89.7979633070
30	93.8560793373	93.0706896015	93.9867190522	92.9395298248

Tables of $j_{\nu,s}$—*Continued*

s	$\nu = \frac{1}{3}$	$\nu = -\frac{1}{3}$	$\nu = \frac{2}{3}$	$\nu = -\frac{2}{3}$
8	25.3907144312	23.2959758671	25.5193219366	23.1625059341
9	28.5327278949	26.4380633449	28.6615845511	26.3049025693
10	31.6746579988	29.5800458826	31.8037149655	29.4471278984
11	34.8165272733	32.7219536520	34.9457487864	32.5892313867
12	37.9583508024	35.8638062730	38.0877098894	35.7312451311
13	41.1001390646	39.0056170547	41.2296148817	38.8731908670
14	44.2418995658	42.1473953433	44.3714756721	42.0150838362
15	47.3836378232	45.2891478947	47.5133010223	45.1569350434
16	50.5253579832	48.4308797136	50.6550975224	48.2987526323
17	53.6670632220	51.5725945862	53.7968702239	51.4405427586
18	56.8087560137	54.7142954298	56.9386230642	54.5823101620
19	59.9504383142	57.8559845286	60.0803591576	57.7240585508
20	63.0921116899	60.9976636963	63.2220809997	60.8657908680
21	66.2337774103	64.1393343918	66.3637906137	64.0075094793
22	69.3754365150	67.2809978012	69.5054896572	67.1492163076
23	72.5170898635	70.4226548994	72.6471795012	70.2909129325
24	75.6587381727	73.5643064946	75.7888612893	73.4326006631
25	78.8003820451	76.7059532629	78.9305359832	76.5742805938
26	81.9420219908	79.8475957737	82.0722043975	79.7159536465
27	85.0836584448	82.9892345106	85.2138672265	82.8576206036
28	88.2252917800	86.1308698863	88.3555250659	85.9992821328
29	91.3669223180	89.2725022556	91.4971784295	89.1409388079
30	94.5085503377	92.4141319250	94.6388277630	92.2825911246

NYMTP

150 Nassau Street
New York City

RECENT MATHEMATICAL TABLES

299[A].—*Mathematical Cuneiform Texts*, edited by O. E. NEUGEBAUER and A. J. SACHS. (American Oriental Series, volume 29.) New Haven, Conn., Amer. Oriental So., 329 Sterling Memorial Library, and the Amer. Schools of Oriental Research, 1945. x, 177 p. + 49 plates. $5.00

In Quellen u. Studien zur Geschichte der Mathematik . . ., A. Quellen, v. 3, (MKT I–III) 1935–37, Otto Neugebauer, the outstanding authority on ancient mathematics and astronomy, published a monumental pioneer work on Mathematical Cuneiform Texts, tablets in collections of Berlin, Brussels, Istanbul, Jena, Leyden, London, New Haven, Oxford, Paris, Philadelphia, Strassburg, and Toronto. Nearly 4000 years ago the Babylonians knew how to find the positive roots of quadratic equations; how to solve simultaneous linear and quadratic equations; and how to calculate compound interest; and they knew that the angle in a semi-circle is a right angle, and our familiar relation between the hypotenuse and sides of a right-angled triangle.

The present work in English is a detailed study of about 200 tablets in the United States[1] and nearly a dozen in Europe. New material of great interest is presented.

In all Babylonian mathematical and astronomical work, tables were fundamental, especially tables for multiplication, squares, square roots, cubes and cube roots, and tables of reciprocals. Since 60 was basic in the sexagesimal notation of the Babylonians, if n were a number written in this notation, $\bar{n} = 1/n$ could be expressed in a finite number of terms (e.g. $\overline{9} = 0;\ 6,\ 40$), or in a never ending series of terms (e.g. $\overline{7} = 0;\ 8,\ 34,\ 17,\ 8,\ 34,\ 17\ \ldots$, with 8, 34, 17 being repeated indefinitely). The necessary and sufficient condition for the first case is that n must be of the form $n = 2^{\alpha}3^{\beta}5^{\gamma}$, where α, β, γ are integers or zero. Such numbers are called regular numbers. In a Louvre tablet dating from about 350 B.C. there are

252 entries in a table of one-place to seventeen-place regular number divisors, and one-place to fourteen-place reciprocals of these numbers.

By all odds the most interesting tablet discussed in the present volume is one at Columbia University, Plimpton 322, dating from 1900–1600 B.C., and containing a table of Pythagorean numbers. If l denote the longer, s the shorter side of a right triangle, and h its hypotenuse, then l, s, and h satisfy the relation

$$l^2 + s^2 = h^2.$$

The values of h and s, always integers, for 15 such Pythagorean triangles are given in two columns of our text; a third column does not give l but h^2/l^2. While the successive numbers in the h and s columns are very irregular, the ratios h^2/l^2 with successive entries decrease almost linearly. The authors discuss the contents of this tablet at great length, and indicate their belief that the following Euclidean relations (*Elements*, book 10, prop. 28, lemma), were known to early Babylonians:

$$l = 2pq, \qquad s = p^2 - q^2, \qquad h = p^2 + q^2,$$

p and q being relatively prime, and $p > q$. Also that the following relation was not only known:

$$h/l = \tfrac{1}{2}(pq + q/p),$$

but was the basis, with known tables of reciprocals, for the discovery of the successive entries of the tablet.

Line 13 of the tablet leads to the 15-fold values of the triangle with sides $l = 4$, $s = 3$, $h = 5$, and in the last line the values for the sides of the resulting triangle, have the common factor 2. All other solutions are relatively prime. For example, line 4 indicates the triangle $l = 13500$, $s = 12709$, $h = 18541$.

The authors point out that final remarks about the character of Babylonian mathematics in MKT III [1937, p. 80] contain the sentence: "Man wird also erwarten können, dass noch eine Art elementarer Zahlentheorie erkennbar wird—etwa so, dass das 'pythagoreisch' der älteren historischen Schule besser 'babylonisch' wird heissen dürfen." "This is fully confirmed," the authors continue, "by the text discussed here. We now have a text of purely number-theoretical character, treating a problem organically developed from other problems already well known and solved by using exactly those tools the development of which is so characteristic for Babylonian numerical methods."

What has been written above is naturally not intended as a review of this very remarkable contribution to scholarship. We were merely drawing attention to certain tables appropriate for consideration in *MTAC*.

R. C. A.

[1] Yale University, 91; University of Pennsylvania 82; Morgan Library 7; Metropolitan Museum 6; Plimpton Library 2; and others at Ann Arbor, Chicago, Ithaca, Princeton and Urbana. The largest and most interesting collection of mathematical tablets at any place in the world is at Yale University.

300[A, K, O].—NYMTP, *Binomial Distribution Functions*, New York, May, 1944, 10 leaves. Hectographed on one side of each sheet. 21.5 × 35.5 cm. A report submitted to the Applied Mathematics Panel National Defense Research Committee. These tables are available only to certain Government agencies and activities.

This report provides values of the function

$$P(n, r, p) = \sum_{k=0}^{r} {}_nC_k p^k q^{n-k}, \qquad p + q = 1,$$

computed over the following ranges of the three variables: n = [5(1)10(5)30(10)50; 6D]; $r = 1(1)(n-1)$; $p = .01(.01).1(.1).5$.

The function finds special application in connection with the binomial frequency distribution in the theory of statistics. Thus, the probability that an event will happen at least r times in n trials is equal to $P(n, r, p)$, where p is the probability that the event will happen in a single trial, and q the probability that it will fail to happen.

It is of some interest to indicate the relationship of the values given in this report and those in the table of E. C. D. MOLINA in *Poisson's Exponential Binomial Limit* (see *MTAC*, v. 1, p. 18), where one finds the tabulation of the function

$$P(c, a) = \sum_{x=c}^{\infty} \frac{a^x e^{-a}}{x!}.$$

It is well known in the theory of probability that if $n \to \infty$, $p \to 0$, while $a = np$ remains constant, then the binomial function $P(n, r, p)$ will approach the limiting value $1 - P(r + 1, a)$. In applications involving small probabilities it has been the general practice to use the function $P(c, a)$ rather than the exact binomial function because of the analytical advantages of the Poisson formulation. But when p becomes fairly large, let us say greater than .1, then the discrepancy between the limit function and the exact function may become statistically significant. The order of the error may be inferred from the following values: When $n = 50$, $p = .01$, $r = 1, 2, 3$, we have for $P(n, r, p)$ the values: .910565, .986183, .998404, while for $a = 50 \times .01 = .5$, $r = 1, 2, 3$, we find for $1 - P(r + 1, a)$ the corresponding values: .908796, .985612, .998248.

H. T. D.

301[D].—HÉCTOR BALANDRANO, CARLOS CABALLERO, GUILLERMO ESTRADA, JORGE LELO DE LARREA, JOSÉ LICHI, GUSTAVO MONTES, MARIO MORALES, & LUIS MORONES, "Tablas numericas para el calculo de radiadares," Seminario de Radiación, Escuela Superior de Ingeniería Mecánica y Eléctrica, Sección Electromagnética, *Comisión Impulsora y Coordinadora de la Investigación Cientifica. Anuario 1944.* Mexico, 1945, p. 125–247. 16.9 × 23 cm.

These three tables calculated by students were as follows:

T.I, p. 128. $\pi l/\lambda$, for $l/\lambda = [.1(.1)5; 6D]$

T.II, p. 129–221. $x = (\pi l/\lambda)(1-\cos\theta)$, and $x = (\pi l/\lambda)(1 + \cos\theta)$, to 5D, for $l/\lambda = .1(.1)5$, $\theta = 0(1°)180°$.

T.III, p. 222–247. (a) $\sin x$, (b) $\cos x$, (c) $\sin x/x$, (d) $\cos x/x$, $x = .001(.001)1$; to 9D for (a) and (b); to 7 or 8S for (c) and (d).

It is stated, that use was made of the 8D NYMTP, *Tables of Sines and Cosines for Radian Arguments*, 1940, but no reference is made to the 9D NYMTP, *Tables of Circular and Hyperbolic Sines and Cosines for Radian Arguments*, 1939 or 1940; or to tables by J. R. AIREY of $\sin x$ and $\cos x$, for $x = [0(.001)1.6; 11D]$, which appeared in B.A.A.S., *Report*, 1916, p. 59–87, and in BAASMTC, v. 1, 1931, and 1946, p. 8–23. T.III, (c) and (d) seem to be new, although partially covered by K. HAYASHI, *Tafeln der Besselschen, Theta-, Kugel- und anderer Funktionen*, Berlin, 1930, p. 30 f. at interval .01.

R. C. A.

302[D, L, M].—SO. OF AUTOMOTIVE ENGINEERS, WAR ENGINEERING BOARD, *The Development of Improved Means for Evaluating Effects of Torsional Vibration on Internal Combustion Engine Installations*, New York, 1945, p. 199–207, 345–347. 21.5 × 28 cm. Offset print.

Table 19, p. 199–206, is of $T(x) = \dfrac{180}{x} \tan x$, for $x = [0(0°.1)360°; 4–5S]$ with average Δ.

Table 20, p. 207, is of $\dfrac{\pi}{180} x \tan x$, for $x = [0(1°)360°; 4–5S]$.

On p. 345, are Tables of $K = \int_0^{\frac{1}{2}\pi} \frac{dx}{(1 - \sin^2 \frac{1}{2}\gamma_0 \sin^2 x)^{\frac{1}{2}}}$, of $(2/\pi)K$, and of $[(2/\pi)K]^2$, to 9D, for $\gamma_0 = 0(5°)180°$.

On p. 346–347, are Tables to 7–8S, of γ, $\sin \gamma$, $-2 \cos \gamma_0 \sin \gamma + \frac{2}{3} \sin 2\gamma$, $(2)^{\frac{1}{2}} \sin \gamma (\cos \gamma - \cos \gamma_0)^{\frac{1}{2}}$, for $\gamma_0 = 10°(10°)160°$, $\tau = 0(10°)90°$, where $\gamma = 2 \sin^{-1} [\sin \frac{1}{2}\gamma_0 \operatorname{sn} (H\tau, \sin \frac{1}{2}\gamma_0)]$, $H = 2K/\pi$. [$H\tau$ is the integral K with β substituted for $\frac{1}{2}\pi$ in the upper limit; hence $\sin \beta = \operatorname{sn} (H\tau, \sin \frac{1}{2}\gamma_0) = \sin \frac{1}{2}\gamma / \sin \frac{1}{2}\gamma_0$.]

303[D, E, K, L].—BAASMTC, *Mathematical Tables*, v. 1, *Circular & Hyperbolic Functions, Exponential & Sine & Cosine Integrals, Factorial Function & Allied Functions, Hermitian Probability Functions*. Second ed. Cambridge, University Press, for the BAAS, 1946. xii, 72 p. 20.9 × 28 cm. 10 shillings. New York, Macmillan & Co., 60 Fifth Avenue, $2.50.

The first edition of these tables was published fifteen years ago and we have had occasion frequently to refer to them in *MTAC*; e.g., v. 1, p. 15, 72, 73, 109, 323, 363. Errors in this edition listed elsewhere in the present issue, MTE 84, were reprinted from a sheet procurable on application at the office of the BAAS, Burlington House, London, W.1. These errors were corrected in the stereotype plates used for printing the Mathematical Tables, p. (1)–(72) of the present edition. All other changes are in the introductory material, the 34 pages of the first edition being here reduced to 7 through elimination of descriptions of the functions tabulated, and bibliographical and historical data. Thus some libraries will naturally try to secure the first edition, with its errata sheet, as well as the second.

The tables in the volume are as follows:

I. Multiples of $\frac{1}{2}\pi$[1(1)100; 15D], computed by L.J.C.
II. Circular sines and cosines, radian arg. [0(.1)50; 15D] c.b. A. T. Doodson, J. R. Airey & L.J.C.
III. Circular sines and cosines, [0(.001)1.6; 11D], c.b. J. R. Airey.
IV. Hyperbolic sines and cosines of πx [0(.0001).01; 15D], δ^2, c.b. L.J.C.
V. Hyperbolic sines and cosines of πx [0(.01)4; 15D], c.b. J. R. Airey.
VI. Hyperbolic sines and cosines of x [0(.1)10; 15D], c.b. J. R. Airey.
VII. Exponential integral [5(.1)15; 10–11S], and Ei $(\pm x) - \ln x$ [0(.1)5; 11D], δ^2, δ^4, δ^6 (some modified), c.b. R. A. Fisher & J. R. Airey.
VIII. Sine and cosine integrals [0(.1)5; 11D], [5(.1)20(.2)40; 10D], δ^2, δ^4, δ^6 (some modified), c.b. R. A. Fisher & J. R. Airey.
IX. Factorial function [0(.01)1; 12D], δ^2, δ^4, (some modified), c.b. R. A. Fisher.
X. Logarithmic factorial integral [0(.01)1; 10D], δ^2 (modified), c.b. G. N. Watson.
XI. Digamma function, $d \ln (x!)/dx$, [0(.01)1, 10(.1)60; 12D].
XII. Trigamma function, $d^2 \ln (x!)/dx^2$, [0(.01)1, 10(.1)60; 12D].
XIII. Tetragamma function, $-d^3 \ln (x!)/dx^3$, [0(.01)1, 10(.1)60; 12D].
XIV. Pentagamma function, $d^4 \ln (x!)/dx^4$, [0(.01)1, 10(.1)60; 10D, 12D] XI–XIV, δ^2, δ^4 (some modified), c.b. A. Lodge & XIV with the cooperation of J. Wishart.
XV. Hermitian probability functions, $Hh_n(x)$, $n = -7(1)+21$, $x = [\geq -7(.1) \leq +6.6$; 6–10D].
XVI. $Hh_0(x)Hh_2(x)/\{Hh_1(x)\}^2$, $x = [-7(.1) + 5$; varies from 9D to 3D]. XV–XVI δ^2, δ^4 (some modified), c.b. J. R. Airey.

One new table by S. Johnston is given on p. xi. This extends each of the digits in the values of the factorial function, up to 18D; but corresponding changes in the differences are not indicated.

The 8 Constants to 16D in the first edition are now increased to 10 by the addition of $\pi/180$ and $180/\pi$.

The new edition was prepared by J. Henderson, L. M. Milne-Thomson, E. H. Neville, & D. H. Sadler.

This is a most admirable volume which ought to be in every college mathematical library. The gilt title along the back of the present edition is a great improvement over the inked title of the old one.

R. C. A.

304[D, M, P].—WERNER F. VOGEL, *Involutometry and Trigonometry*. Detroit, Mich., Michigan Tool Co., 7171 McNichols Rd., 1945. xii, 321 p. 19.5 × 27.2 cm. $20.00. Procurable only by direct application to the publisher.

This volume has been prepared primarily for the gear and tool engineer. In the words of the author, involutometry "deals with the involute of a circle and with surfaces containing a multitude of involutes." The volume comprises a Main Table and three appendices. The Main Table (p. 1–181) gives the values of the six circular functions and the two "involumetric" functions, $\text{inv}\,\phi = \tan\phi - \text{arc}\,\phi = \text{arc}\,\theta$ and $\text{coinv}\,\phi = \text{inv}\,(90° - \phi)$. These eight functions are tabulated to 7D, for $\phi = 0(0°.01)90°$. Two additional columns give the values in radians of arc ϕ (tabulated to 7D on odd-numbered pages), and the values of θ in degrees and decimals of a degree (tabulated on even-numbered pages). The first differences are tabulated alongside the corresponding functions, except for the column of arc ϕ whose differences are noted at the bottom of the pages.

For purposes of interpolation, a one-page critical table gives the values of $\phi° \cot\phi$ and appropriate values of $\cot\phi$, as well as the values of $\phi° \csc\phi$ and appropriate values of $\csc\phi$ for very small arguments.

Two other involumetric functions defined by the author are $\text{ev}\,\theta = \sec\phi$ and arc $\epsilon = \tan\phi$. The variables ϕ, θ and ϵ are referred to by the gear engineer as the pressure, polar and roll angle of the involute, respectively. Various types of problems involving these angles as well as the radius vector $r = r_b\,\text{ev}\,\theta$ and the radius of curvature $\rho = r_b \tan\phi$, where r_b is the radius of the base circle, may be readily solved with the aid of the Main Table. Several illustrations of such problems are given by the author in Appendix C (p. 298–321). This appendix is devoted to an exposition of the salient points of involutometry such as the generation of the involute of a circle and its geometric properties, the definition of the involumetric functions and a description of curves related to the involute. Among other topics briefly sketched are: *Solid Involutometry* dealing with surfaces containing a multitude of involutes (a typical example is the surface of revolution generated by the rotation of an involute about a given axis) and *Spherical Involutometry* dealing with the involute defined as "the path of any point on the surface of a base cone, when the surface is unwrapped from the base cone and held tightly during this procedure."

Appendix A (p. 189–240) consists of

(a) Various conversion tables (from radians to degrees and decimal fractions of the degree, from degrees and decimal fractions of the degree to radians, etc.)

(b) Polygon tables $\left(\text{values of } \frac{180°}{N}, \sin\frac{180°}{N}, \cos\frac{180°}{N}, 1 - \cos\frac{180°}{N}\right)$ for $N = 1(1)300$. Compare *MTAC*, v. 1, p. 312–313.

(c) Collections of formulae and integrals of trigonometry and involutometry.

(d) A set of involute tracing charts, involute locating charts and cycloid tracing tables.

Appendix B (p. 241–296) consists of a collection of numerical diagrams, tables and charts required in gear calculations.

The tabular material of the six circular functions is based on the tables of BRIGGS in his *Trigonometria Britannica*, Gouda, 1633. Seven-place values of the sine, cosine, tangent and cotangent at interval 0°.001, based on the same source, have also been compiled by J. PETERS (1918) and made available in this country (1942; see *MTAC*, v. 1, p. 12–13). The volume under review possesses the pleasant feature of exhibiting all the circular functions as well as the other four functions above mentioned on two pages facing each other. The

usefulness of the volume is further enhanced by the happy choice of types, as well as the excellent charts and diagrams. The author and publishers are to be congratulated on the meticulous planning of this volume which manifests a constant endeavor to place at the disposal of the tool engineer a complete and lucid exposition of the mathematical foundation for his art, as well as an extremely useful table for the solution of his problems.

ARNOLD N. LOWAN

EDITORIAL NOTES: Attention may be also directed to *MTAC*, v. 1, p. 88–92, where there is discussion of an 8-place table, by EARLE BUCKINGHAM, of $\sin \phi$, $\cos \phi$, $\tan \phi$, $\cot \phi$, $\phi = 0(0^{\circ}.01)90^{\circ}$; and inv. ϕ at interval $0^{\circ}.01$ up to 60°; 12D are given up to $0^{\circ}.5$, 10 to 1°, 8 to 37°, and 7 to 60°.

Those inquiring as to the originality of Mr. Vogel's work should compare it with J. T. PETERS, *Sechsstellige Werte der Kreis- und Evolventen-Funktionen von Hundertstel zu Hundertstel des Grades nebst einigen Hilfstafeln für die Zahnradtechnik*, Berlin and Bonn, 1937. For example, except for 7-place tables, instead of 6-place tables, p. 2–181 of Vogel are identical in every respect, including order of columns, spacing, face of type, and rules, with p. 2–181 of Peters; the auxiliary interpolation table of p. 182 of Vogel is not expanded at all, but is identical with that of Peters.

In reviewing a previous publication of W. F. Vogel (*MTAC*, v. 1, p. 312) a statement of Peters was interpreted as suggesting that Vogel was a former member of the computing staff of the Rechen-Institut. We are informed that this inference was incorrect.

305[F, Z].—RAPHAEL M. ROBINSON, *Stencils for Solving* $x^2 \equiv a \pmod{m}$. Berkeley and Los Angeles, Univ. of California Press, 1940. 272 stencils with a booklet of 14 p.

This set of stencils for solving quadratic congruences appeared in 1940 just when the reviewer's *Guide to Tables in the Theory of Numbers* was in page proof so that it received there only the briefest attention. With the present or pending "reconversion" of many number-theorists from war work it may be of interest to call attention to this tool for the solution of a problem familiar to such computers.

The most general quadratic congruence, as is well known, may be reduced to solving for x the congruence

$$x^2 \equiv a \pmod{m} \tag{1}$$

or the equivalent diophantine equation

$$x^2 = a + my. \tag{2}$$

According to Gauss' method of exclusion a convenient set of excluding numbers E is used to determine y by considering (2) modulo E. Those values of y modulo E which make $a + my$ a quadratic non-residue (mod E) are clearly impossible. Thus for each E, y is restricted to a set of about $\frac{1}{2}E$ arithmetic progressions, each with the difference E.

The present stencils offer a rapid and convenient way of finding the one or two values of y which are common to these sets of arithmetic progressions. For each E, the possible values of y are represented by holes punched in a standard IBM card. To find y in (2), and hence x in (1), the computer simply selects nine cards (corresponding to $E = 5, 7, 9, 11, 13, 16, 17, 19, 23$) from the set of 272 cards depending on the values of a and m modulo E. Superposing the cards gives at a glance the values of y common to the cards, and hence those y's for which (2) has a chance of a solution. Each of these is then tried in (2) by comparing the corresponding values of $a + my$ with a table of squares. In this way quadratic congruences with moduli < 3000 are easily solved. The limit is set by the facts that $y < m/4$, and the capacity of the IBM card allows $0 \leq y < 750$. By using the stencils n times, however, congruences for $m < 750(n + 1)^2$ may be solved, as explained in the booklet accompanying the stencils.

This is the second use so far for IBM cards as a coincidence mechanism, the first being that in the revised edition of Lehmer's *Factor Stencils* by J. D. ELDER.[1] In each case the cards are heavily punched in a way never contemplated by orthodox IBM usage, but in the Robinson stencils the holes follow a regular pattern modulo E so that a simple process of

repeated offset gang punching and comparing by the reproducer produced and checked the whole set of 2E cards from a single card having simply the multiples of E punched. Besides being a useful tool the set of stencils serves as model for other sets for the solution of a wide variety of similar problems.

D. H. L.

[1] D. N. LEHMER, *Factor Stencils, Revised and extended* by J. D. Elder, Washington, Carnegie Institution, 1939, 27 p. + 2135 stencils.

306[J, L, M].—L. S. GODDARD, "On the summation of certain trigonometric series," Cambridge Phil. So., *Proc.*, v. 41, part 2, Aug. 1945, p. 145–160. 17.3 × 25.6 cm.

The following functions, (a)–(d), are tabulated, $n = [1(1)10; 4D]$:

(a) $-S_n^1(\alpha) = \alpha^{-1}\int_0^\pi \sin^2 nx \cot(x/\alpha)dx$, $\alpha = 1(.25)1.5(.5)3(1)8, \infty$; for $\alpha = \infty$, $-S_n^1(\alpha) = \frac{1}{4}\operatorname{cin}(2\pi n) = \frac{1}{2}\int_0^{2\pi n}\frac{1-\cos x}{x}dx$.

(b) $T_n^1(\alpha)/n = (2\alpha)^{-1}\int_0^\pi (\pi - x)\sin 2nx \cot(x/\alpha)dx - \frac{1}{4}\pi^2\alpha$ (the term $\frac{1}{4}\pi^2\alpha$ being omitted when n is integral but αn is non-integral), $\alpha = 1(.25)1.5, 2, 4, 8, \infty$; for $\alpha = \infty$, $T_n^1(\alpha)/n = \frac{1}{2}\pi\int_0^{2\pi n}\frac{\sin x}{x}dx$.

(c) $-S_n^2(\alpha) = -\alpha^{-2}n^{-2}S_n^1(\alpha) + \theta_3(\alpha)$, $\alpha = 1(.25)1.5(.5)3(1)8$; $\theta_3(\alpha) = \sum_{m=1}^{\infty} m^{-3}\sin^2(\pi m/\alpha)$.

(d) $T_n^3(\alpha) = \alpha^{-2}n^{-2}[T_n^2(\alpha) - S_n^1(\alpha)]$, α as in (c).

There are also tables to 4D of

(e) $-S_n^2(\alpha) = -\pi^2(\alpha - 1)/2\alpha^2$; (f) $T_n^2(\alpha) = \pi^2/(4\alpha)$, (g) $\alpha^2\theta_3(\alpha)$, for α as in (c).

Following S. A. SCHELKUNOFF, *Quart. Appl. Math.*, v. 2, 1944, p. 90, Goddard uses the notation $\operatorname{cin} x = \int_0^x \frac{1-\cos t}{t}dt$, p. 157.

R. C. A.

307[K].—H. O. HARTLEY & M. SUMNER, "Tables of the probability integral of the mean deviation in normal samples," *Biometrika*, v. 33, Nov. 1945, p. 252–265. 19.3 × 27.4 cm. These pages are occupied as follows: "The probability integral of the mean deviation, editorial note" by E. S. PEARSON, p. 252–253; "On the distribution of the estimate of mean deviation obtained from samples from a normal population" by H. J. GODWIN, p. 254–256; "Appendix, Note on the calculation of the distribution of the estimate of mean deviation in normal samples" by H. O. HARTLEY, p. 257–258; tables, p. 259–265.

Suppose $x_1, x_2, \cdots, x_n$ are the values of x in a sample from a normal population in which the distribution of x is $(2\pi)^{-\frac{1}{2}}e^{-\frac{1}{2}x^2}$. The mean deviation m of the sample is defined by

$$m = n^{-1}\sum_{i=1}^{n}|x_i - \bar{x}|, \qquad \text{where } \bar{x} = n^{-1}\sum_{i=1}^{n}x_i.$$

GODWIN has shown that the probability element $f_n(m)dm$ for m is given by

$$(1)\quad f_n(m)dm = \frac{n^{\frac{1}{2}}}{2^{\frac{1}{2}(n+1)}\pi^{\frac{1}{2}(n-1)}}\left\{\sum_{k=1}^{n-1}\binom{n}{k}\exp\left[-\frac{m^2n^3}{8k(n-k)}\right]G_{k-1}(\tfrac{1}{2}nm)G_{n-k-1}(\tfrac{1}{2}nm)\right\}dm$$

where $G_r(x)$ is defined by

$$G_r(x) = \int_0^x \exp\left[-\frac{t^2}{2r(r+1)}\right] G_{r-1}(t)dt, \tag{2}$$

and $G_0(x) = 1$.

HARTLEY has tabulated

$$\int_0^m f_n(m)dm \tag{3}$$

for $n = [2(1)10; 5D]$, and for $m = 0(.01)3$, by a recurrence procedure of numerical quadratures. The starting point of the recurrence was the table of

$$\pi^{-\frac{1}{2}}G_1(x) = \pi^{-\frac{1}{2}}\int_0^x e^{-(\frac{1}{2}t)^2}dt = 2\pi^{-\frac{1}{2}}\int_0^{\frac{1}{2}x} e^{-\alpha^2}d\alpha, \tag{4}$$

given in NYMTP, *Tables of the Probability Integral*, v. 1, 1941. This integral was multiplied by $2\pi^{-\frac{1}{2}}e^{-(\frac{1}{2}x)^2}$ which is also tabulated in the NYMTP *Tables*. Products were formed at interval .05 in x. The function $480\pi^{-\frac{1}{2}}2^{-\frac{1}{2}}G_2(x)$ was obtained in accordance with (2) by numerical quadrature; the factor 480 was introduced because of the particular method of quadrature used. This process was repeated for $r = 3(1)8$, thus producing the functions $G_r(x)$. By writing

$$g_r(x) = G_r(x)\cdot\exp\left[-\frac{x^2}{2(r+1)}\right] \tag{5}$$

where $x = \frac{1}{2}nm$, the expression (1) may be rewritten as

$$f_n(m) = n^{\frac{1}{2}}2^{-\frac{1}{2}(n+1)}\pi^{-\frac{1}{2}(n-1)}\sum_{k=1}^{n}\binom{n}{k} g_{k-1}(x)g_{n-k-1}(x). \tag{6}$$

The functions $g_r(x)$ were obtained by the multiplication indicated by (5), and the computation of the integral (3) was finally made by using expression (6).

A table of values of m_l (lower percentage points), for which

$$\int_0^{m_l} f_n(m)dm = .001, .005, .01, .025, .05, .1,$$

is given for $n = 2(1)10$; also a table of values of m_u (upper percentage points), for which

$$1 - \int_0^{m_u} f_n(m)dm = .001, .005, .01, .025, .05, .1,$$

is given for $n = 2(1)10$.

S. S. W.

308[L].—H. BATEMAN, "Some integral equations of potential theory," *J. Appl. Physics*, v. 17, Feb. 1946, p. 91–102. 20 × 26.8 cm.

Incidentally there is a table (p. 98–99) of $P_n(1 - 2e^{-t})$, for $n = 1(1)10$, $t = [1(1)20; 15D]$, where $P_n(x)$ is the polynomial of Legendre of order n. Quotation: "It is thought that this and similar tables may be useful for the tabulation of functions defined by definite integrals, which are hard to compute directly but have simple Laplace transforms."

309[L].—V. FOK, "Raspredelenie tokov, vozbuzhdaemykh ploskoĭ volnoĭ na poverkhnosti provodnika" [Distribution of currents induced by a plane wave on the surface of a conductor], Akademiîa Nauk, SSSR, *Zhurnal Eksperimental'noĭ i Teoreticheskoĭ Fiziki*, v. 15, no. 12, Dec. 1945, p. 700–701. 16.5 × 25.8 cm.

$g(x) = \int e^{(i-1)tx+\frac{1}{2}it^2}dt$, $G(x) = e^{\frac{1}{2}ix^2}g(x)$. Re G, Im G, and $|G|$ are given for $x = [-4.5(.1) - 3.5; 4D]$, $[-3.4(.1)1; 3D]$; arc G is given to the nearest 1″ to −3.5, and to the nearest 1′ thereafter. Further, Re g, Im g, and $|g|$ are given for $x = [-1(.1)2.4; 3D]$, $[2.5(.1)4.5; 4D]$, and arc g is given to the nearest 1′.

310[L].—J. C. JAEGER, "On thermal stresses in circular cylinders," *Phil. Mag.*, s. 7, v. 36, 1945, p. 419–423. 17.1 × 25.5 cm.

Consider the thermal stresses in the cylinder $0 \leq r < a$, initially at constant temperature V, whose surface is kept at zero temperature for $t > 0$. If E is Young's modulus, ν Poisson's ratio, α the coefficient of linear expansion of the solid, v its temperature at radius r, K, ρ, c the thermal conductivity, density, and specific heat of the material of the cylinder, and $\kappa = K/\rho c$; and if, finally σ_r and σ_θ are the radial and tangential stresses, then

$$(1-\nu)\sigma_r/E\alpha V = 2\sum_{s=1}^{\infty} \frac{e^{-\kappa\beta_s^2 t/a^2}}{\beta_s^2 J_1(\beta_s)}\left\{ J_1(\beta_s) - \frac{a}{r} J_1\left(\frac{r\beta_s}{a}\right)\right\}$$

$$(1-\nu)\sigma_\theta/E\alpha V = 2\sum_{s=1}^{\infty} \frac{e^{-\kappa\beta_s^2 t/a^2}}{\beta_s^2 J_1(\beta_s)}\left\{ J_1(\beta_s) + \frac{a}{r} J_1\left(\frac{r\beta_s}{a}\right) - \beta_s J_0\left(\frac{r\beta_s}{a}\right)\right\}$$

where $\pm\beta_s$, $s = 1, 2, \cdots$, are the roots of $J_0(\beta) = 0$. There are tables of (i) $(1-\nu)\sigma_r/E\alpha V$ and (ii) $(1-\nu)\sigma_\theta/E\alpha V$ for $\kappa t/a^2 = [.005(.005).02(.01).1(.05).2(.1)1; 4D]$, $r/a = 0(.1).7(.05).95$; and in the case of (ii) also for $r/a = 1$. On p. 423 are corresponding graphs of (i) and (ii) for $0 \leq \kappa t/a^2 \leq .1$.

311[L].—E. L. KAPLAN, "Auxiliary table of complete elliptic integrals," *J. Math. Phys.*, v. 25, Feb. 1946, p. 26–36. 17.4 × 25.4 cm.

The integrals in question are

$$K = \int_0^{\frac{1}{2}\pi} (1 - k^2 \sin^2\phi)^{-\frac{1}{2}} d\phi, \qquad E = \int_0^{\frac{1}{2}\pi} (1 - k^2 \sin^2\phi)^{\frac{1}{2}} d\phi.$$

The table (p. 29–36) gives values of K and E as functions of $\log k'^2 = -\lambda$, for $\lambda = [1(.005)2(.01)6; 10D]$, with $\mu\delta^2$ or $\frac{1}{2}\mu\delta^2$, where $k'^2 = 1 - k^2$.

The only comparable table which facilitates the getting of values of K and E in the neighborhood of their singularity is that of J. R. AIREY, "Toroidal functions and the complete elliptic integrals," *Phil. Mag.*, s. 7, v. 19, 1935, p. 180–187. He has tabulated to 10D functions K_1, K_2, E_1, E_2, such that

$$K = K_1 \ln(4/k') - K_2, \qquad E = E_1 \ln(4/k') + E_2.$$

For each K and E there are three quantities to be obtained from tables, and some arithmetical operations to be done besides. The importance of these functions makes it worthwhile to provide a more direct means of getting their values for small values of k'. With the present table the only preliminary to extracting the value desired is the determination of $\log k'^2$. The tabular argument is given as a logarithm with a negative integral part and a positive decimal, in order that the user of the table may be spared the trouble of getting the cologarithm when k'^2 is given as a decimal.

The method of the arithmetico-geometric mean, followed by subtabulation to one-tenth of the original interval, was used for $1 \leq \lambda \leq 2$, while the rest of the table was computed by means of auxiliary functions like Airey's. In both cases the check was by differencing. The use of both methods for $\lambda = 2$ is a check on errors of procedure. From eleven to fourteen decimals were retained.

The most useful general tables of K and E are those of K. HAYASHI (*Tafeln der Besselschen, Theta-, Kugel- und anderer Funktionen*, Berlin, 1930, p. 72–81) who gives the values for $k^2 = [0(.001)1; 10D]$. When k^2 is near unity, however, it is difficult or impossible to interpolate in these tables—hence the need for some kind of auxiliary table. Since Hayashi's tables in general are notorious for their inaccuracy, the writer checked these by differencing (generally every third one of the fourth differences) either the values themselves or auxiliary functions calculated therefrom. They were also compared with the table of L. M. MILNE-THOMSON ("Ten-figure table of the complete elliptic integrals," London Math. So., *Proc.* s. 2, v. 33, 1930, p. 162–163), for $k^2 = [0(.01)1; 9D]$, which served only to show that the last

figure in Milne-Thomson's table is unreliable. The following errors were found in Hayashi's table (only the four starred values were actually recomputed):

		K				E	
k^2	Decimals	For	Read	k^2	Decimals	For	Read
.999	8–9	06	60*	.052	9–10	45	55*
.013	8–9	68	86	.939	9	4	3
.095	0	0	1	.936	9	9	8
.737	9–10	56	65*	.201	3–4	65	86
.493	9	4	9	.732	10	6	8*
				.668	0	0	1
				.669	The first two digits of the argument should be given.		

The eleventh and twelfth decimals given in a portion of the table were not checked, nor were any of the rather inadequate auxiliary tables.

Extracts from introductory text

312[U].—E. E. BENEST & E. M. TIMBERLAKE, *Astro-Navigation Tables for the Common Tangent Method specially developed and arranged.* Cambridge, Heffer, 1945. 107 p. 21.5 × 27.8 cm. 7s. 6d.

These tables were prepared for use with the common tangent method, a method recommended, as the authors point out, a number of years ago by RADLER DE AQUINO. They do not mention the recent excellent *Astronomical Position Lines Tables* of JUAN GARCÍA (RMT 272) which are intended for use with the same method.

The tables are somewhat similar in content and arrangement to those in WEEMS' *Line of Position Book* (RMT 298). However instead of determining an altitude and an azimuth for a single assumed position, two altitudes are computed for adjacent integral latitudes and the line of position is drawn as one of the common tangents to two circles with centers at the two assumed positions and with radii equal to their respective altitude intercepts. Six or seven "figures" are given in the logarithmic values (corresponding to 10^5 or 10^6 times five- or six-place logarithms of secants and cosecants with enough zeros placed at the left in each case to provide six or seven "figures"), and it is stated that "In general: 5 figures will give an altitude to an accuracy of 2′, 6 figures will give an altitude to an accuracy of 1′, 6 figures, with interpolation, will give an altitude to an accuracy of 0′.1."

The astronomical triangle is divided into two right triangles by a perpendicular dropped from the zenith upon the hour circle of the celestial object. K is the latitude of the foot of the perpendicular, or more precisely, it is the angle from the celestial equator to the foot of the perpendicular, measured toward, and if necessary, through the elevated pole. A is usually $10^5 \log \sec a$ where a is the length of the perpendicular mentioned above. For $LHA < 11°$, $L < 38°$, and for a few values for $LHA = 11°, 12°$, $A = 10^6 \log \sec a$ and values are given to seven "figures."

The formulae usually employed are: $A = 10^5 \log \sec a$; $B = 10^5 \log \sec (K \sim d)$; $C = 10^5 \log \csc h = A + B$.

The first table (**A** and **K**) covers 60 pages. The vertical argument is latitude = 0(1°)90° given in two columns on facing pages, 0 to 45° on the left and 45° to 90° on the right. Three integral values of local hour angle appear on each such pair of pages with the single exception that the first page contains only all of the values corresponding to zero latitude. On page 6 in the explanation, it is erroneously stated, "latitudes 0° to 45° always on the left-hand page, and . . . ." In addition to the usual headings corresponding to local hour angle = 0(1°)90°, there are also other headings on each page corresponding to values of the local hour angle in the other three quadrants. The tabulated quantities in this table are **K** as defined above given to the nearest tenth of a minute of arc in two columns corresponding to the local hour angle in the first or fourth quadrant, and second or third quadrant. **A**

is given to six or seven "figures" with the sixth or seventh set off to the right for the convenience of the air navigator who may wish to use only five or six.

In **Table B** which covers 17 pages, the horizontal argument is $K \sim d = 0(1°)90°$, six values to a page except at the end, and the vertical argument is the integral minutes of $K \sim d$. The tabulated quantity is B, $10^5 \log \sec (K \sim d)$ given to six figures with the sixth set off, for $K \sim d$ 10° to 90° and $10^6 \log \sec (K \sim d)$ given to seven figures with the seventh set off, for $K \sim d$ 0 to 10°. In addition, to the right of each column of values of B is a column giving proportional parts (1 to 9 tenths) of the tabular differences, making interpolation very simple. For values of $K \sim d$, 86° to 89° inclusive, special interpolation tables giving 1 to 99 hundredths of the tabular differences are given, but even with these, an uncertainty of as much as 100 units in the last place given may exist in the interpolated value; even so, the uncertainty in the calculated altitude will usually be of the order of a tenth of a minute of arc or less.

Table C covers ten pages and has as its horizontal argument, the integral degrees of altitude = 0(1°)90°, 8 values to a page. The vertical argument is C, $10^5 \log \csc h$ given to six figures with the sixth set off, for h 0 to 80°; $10^6 \log \csc h$ given to seven figures with the seventh set off, for 80° to 90°. This arrangement by which logarithms are sometimes multiplied by 10^5 and sometimes by 10^6, and in which sometimes the sixth figure is set apart, sometimes the seventh, appears to be needlessly confusing. It would be better if a few decimal points were used, and if the figures set aside were always the sixth, or the sixth and the seventh.

The final table is one of four pages for transferring position lines in high latitudes of the northern and southern hemispheres. A brief explanation is given of its use.

On page 5 in the explanation, the statement is made that "The plot in Fig. 2 shows the two possible ways of plotting an almost E./W. position line for the body near the meridian," whereas this figure is simply a repetition of Fig. 1 with letters replaced by numbers, and appears not to be the plot the authors had in mind. Such careless errors as have been noted in the explanation, leave one with a feeling that the tables have been hastily prepared and perhaps cannot be relied upon. As a check on the numerical values, those of A given for hour angles 78°–89° inclusive (1080 values) have been compared with a corrected copy of DREISONSTOK (H.O. 208, RMT 103); 96 errors, each of a single unit in the last place given, were found. If the sample is characteristic of the tables, the numerical values are almost as good as those in COMRIE'S *Hughes' Tables for Sea and Air Navigation* (RMT 115).

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313[U].—GEORGE G. HOEHNE, *Practical Celestial Air Navigation Tables. Volume II, Latitudes 20° to 39° North inclusive.* Miami, The Navigation Publishing Co., 1943, 256 p. 15 × 24.3 cm. Not available for purchase.

This volume is the only one of a projected set of fourteen to appear. There were to have been nine volumes, each containing altitudes and azimuths for twenty-three navigational stars and covering twenty degrees of latitude in a specified hemisphere, except the last volume which was to cover the regions 80° to 89°, North and South. Five other volumes were to have been provided for the sun, moon, planets and such navigational stars as occurred in the zone of declinations, −30° to +30°; these were to have been arranged for declinations of the same and contrary name to the latitude, and hence could have been used in either hemisphere. Since this volume has been removed from the market because of copyright difficulties (see below), it does not seem likely that the remaining thirteen volumes will ever appear.

The principal tables consist of twenty sections of twelve pages each, one for each of the twenty integral latitudes covered. A thumb index cut into the edge of the volume allows one to find quickly the material corresponding to a particular latitude. The vertical argument is T, Local Hour Angle of the March Equinox (or Local Sidereal Time) [0, 1°, 360°]. Each column is headed by the names of one or two stars. Occasionally a column is interrupted to

allow a new star to replace one which has reached a position where it can no longer be satisfactorily observed; the name of the second star appears on the second line at the top of the column and again below where the tabular material is interrupted. Each pair of facing pages carries ten columns and covers 60° of the Local Hour Angle of the March Equinox. The tabulated quantities are altitude plus refraction for an elevation of five thousand feet above sea level, the sum rounded off to the nearest minute, and azimuth (not the usual *azimuth angle*), to the nearest degree.

The Preface indicates that this arrangement is original with the author; according to G. A. PATTERSON of the U. S. Hydrographic Office, it was first suggested by two Americans at about the same time, HOEHNE and HUTCHINGS. It is a very practical arrangement since it brings together on a single pair of facing pages, all of the material likely to be of use to a navigator at a given time and place, and since it eliminates the need for a "star finder" or a similar device. It is true that two entries are required for a given altitude, latitude and star, and one for the star rising and another for it setting. This does not seem very serious when it is realized that there is one less chance for an error of judgment inasmuch as this arrangement allows azimuth to be tabulated instead of azimuth angle. Probably the chief weakness of the tables is that it will be necessary to recompute them from time to time, due to the effects of precession on star positions. Hoehne has partially met this difficulty by two ingenious devices. He has provided a table on the inside of the front cover, "Correction due to Annual Change in Declination to Tabulated Altitude of Stars," and another, Table VII, "Correction to GHA Aries due to Change in Right Ascension of Stars."

Twenty-three stars are listed in this volume, Aldebaran, Alpheratz, Alphecca, Altair, Antares, Arcturus, Betelgeuse, Capella, Caph, Canopus, Deneb, Denebola, Dubhe, Etamin, Fomalhaut, Pollux, Procyon, Rasalague, Regulus, Rigel, Sirius, Spica, Vega. Not every star appears at each latitude; for example, Etamin appears for latitude 31° but not for 35°. Comparing this series of stars with those which appear in the *Astronomical Navigation Tables* (see *MTAC*, v. 1, p. 82 f, hereafter called the *ANT*), of which the American copy is called H.O. 218, it is found that Alphecca, Caph, Denebola, Etamin and Rasalague do not appear in the *ANT*, and that Achernar, Acrux, Peacock and Rigil Kent, which do appear in the *ANT*, are not included in this volume of Hoehne, presumably since it is designed for northern latitudes only.

In the Preface, the following statement occurs:

"The scheme of tabulating the precomputed altitudes and azimuths for integral degrees of Local Hour Angle Aries instead of local hour angle for the stars was conceived and designed by the author and made possible to publish by extracting and interpolating for declination the values of altitude and azimuth from Hydrographic Office Publication 214."

Because of the copyright difficulties mentioned above, it seemed desirable to compare altitudes as given in Hoehne and *ANT* and as they would be interpolated from H.O. 214 (see *MTAC*, v. 1, p. 81 f). Since the values of the altitudes for $d = 45°$, same name as latitude, given in H.O. 214, v. 4, had recently been recomputed, the star Deneb ($d = 45° 04'$) was used as a basis of comparison. All values tabulated in Hoehne for this star for latitude 31° and 35° were compared with values interpolated in H.O. 214 and corrected for refraction according to the table used in the construction of the *ANT*. Of the 16 values for $L = 31°$ and 12 for $L = 35°$ [$d = 45°$, same name], in H.O. 214 known to be in error by two or more units in the last place given, six give values which, when rounded off to the nearest minute, would be one minute in error. In every one of these cases, Hoehne gave the value given in *ANT*, not the erroneous value obtained from H.O. 214. In fact, no differences were found between Hoehne and the *ANT* in 1052 values compared while in a series of 261 values, 25 discrepancies were found between Hoehne and values interpolated from H.O. 214 for stars found in the *ANT*. On the other hand, 75 altitudes for Rasalague, $L = 35°$N, were computed by H.O. 214 and compared with the corresponding values in Hoehne; there are no altitudes for Rasalague in the *ANT*. There were only three discrepancies in the 75 values, each corresponding to a rounding-off "error" resulting from a difference of only 0'.1 in the interpolated value. This is about what would be expected as a result of the interpolation being carried out by two different persons. The evidence presented above seems to indicate

that Hoehne copied most of his altitudes from *ANT* and actually interpolated from H.O. 214 only the relatively few altitudes appearing for the five stars mentioned above which do not appear in the *ANT* and such altitudes of other stars as fall between 5° and 10°. This is serious since the *ANT* are copyrighted and the American copy, H.O. 218, restricted.

It seems most unfortunate that an idea as basically sound and as ingeniously implemented as Hoehne's should have to be shelved. It is to be hoped and expected that a new table, with values interpolated from H.O. 214, or preferably computed afresh, will appear in the near future.

In addition to the tables mentioned above, there are the usual auxiliary tables, arc to time, time to arc, etc.

CHARLES H. SMILEY

314[U].—G. F. MARTELLI, *Martelli's Navigational Tables*, Glasgow, D. M'Gregor & Co., 1946, xi, 50 p. 14.5 × 24.3 cm., 7 shillings and 6 pence.

This is a new edition of a set of tables that has been in use in surface navigation for more than seventy years, principally by British navigators. In 1873, the "Lightning Printing Office," New Orleans, published an edition of *G. F. Martelli's Tables of Logarithms, a simple and accurate method for finding the Apparent Time;* the volume reviewed here has almost the same number of pages and is about the same size. The preface in the 1873 edition, signed by the author, states that "This work was completed in 1853. I now submit it to the seafaring community trusting that its simplicity and accuracy will obtain for it a universal welcome. . . ." It is uncertain whether an earlier edition had been published. The examples worked in the Lightning edition were for 1863 and 1864 and there were letters of commendation for the method signed by Lt. M. F. Maury [1806–1873], U.S.N., and Lt. R. Maxwell, U.S.N., which suggest that these gentlemen had had an opportunity to examine the tables and to try them out. To date no trace of the personal history of G. F. Martelli has been found by the reviewer.

It is interesting to note that, although the tables have been reviewed very unfavorably on several occasions, they have nevertheless been used widely. A "revised edition with additional tables," published by D. M'Gregor and Co. in 1932, still carried examples worked out for 1888. Except for a slight extension of Table 1 (from 71° 35′ to 90° inclusive), improved tables for refraction, correction for sun, planets and stars, and modern examples following current British usage, the 1946 edition is practically identical with that of 1932.

The tables were designed to provide a rapid solution of the spherical triangle based on a slightly modified form of the well-known haversine-cosine formula,

$$\text{hav } z = \text{hav } (L \sim d) + \cos L \cos d \text{ hav } t,$$

where z is the zenith distance of the celestial body; L and d are the latitude of the observer and the declination of the body respectively; t is the meridian angle of the body. Zenith distance (and hence altitude) are said to be determined to the nearest minute of arc.

Table 1, Log. of Lat. and Declination, gives the values of $10^4(.5 + \log \cos x)$ to the nearest integer for $x = 0(1')90°$. In **Table 2, Sum or Difference,** the argument is $x = 0(0.'1)90°$. The tabulated quantity is $10^4(1.2 - 2 \text{ hav } x)$, disguised as a time interval by the use of 0ˢ.1 as the unit; thus for $x = 0$, the tabular value would be, not 12000 but $20^m0^s.0$. In Tables 3 and 4, the same device is used. In **Table 3, Angle of Altitude,** the argument is altitude, $h = 0(1')90°$, and the tabulated quantity is $2(10^4) \text{ hav } z$ where z is the zenith distance of the observed body. In **Table 4, Auxiliary Logarithms,** the argument in "minutes and seconds" is $10^4(1.2 + 2 \text{ hav } x)$ for $20^m0^s(0^s.1)36^m59^s$, and the tabulated quantity is log (1.08/hav x), to 4D. In **Table 5,** the argument is hour angle, $t = 16^h(5^s)7^h$, and the tabulated quantity is log (10.8/hav t) given to 4D.

It will be seen that the haversine-cosine formula can be written

$$(10.8/\text{hav } t) = \sqrt{10} \cos L \sqrt{10} \cos d \frac{1.08}{\text{hav } z - \text{hav } (L \sim d)}$$

and the logarithm of the fraction on the right evaluated by the use of Tables 2, 3, and 4. Table 1 gives the value of log ($\sqrt{10}\cos z$) and Table 5 allows one to find t from the fraction on the left.

The examples of the determination of a position line are carried out in detail for three distinct methods: the longitude method, intercept method, and latitude method. In the first of these, a latitude is assumed and the corresponding longitude on the line of position is computed. The second is the well-known Marcq St. Hilaire method, depending on the difference between a measured altitude and a computed altitude. In the third, a longitude is assumed and the corresponding latitude computed. The azimuth is always computed the same way, using Table 1 only. In each case, the resulting three lines of position, obtained by the three methods, are plotted on a large scale chart and are shown to be essentially the same. Examples are given of the computation of the deviation of a magnetic compass, the amplitude of a rising (or setting) body, the identification of stars, distance and initial course along a great circle and position of the vertex of a great circle.

To determine the probable accuracy of the tabulated values, 1000 values in Table 1 were checked, and 10 errors, each of a unit in the last place due to rounding off, were found. In Table 2, 19 rounding-off errors were found in 500 tabular values, and in Table 3, 16 rounding-off errors were found in 500 values. Since the tables are essentially based on four-place logarithms, it would seem desirable that rounding-off errors be eliminated from the tables, or at the very least, the rounding-off be not done systematically the same way throughout, as appears to be the case in some of the tables.

Remembering that the tables as originally devised did not allow the use of declinations or latitudes greater than 71°34′ and hence avoided negative values in Table 1, one must admire the ingenuity of Martelli in providing tables which have served navigators so long and so well. There would be an advantage, as Browning E. Marean has pointed out, in having an American edition of these tables with explanations designed for use with GHA of sun, moon, planets and star, as tabulated in the *American Nautical Almanac* and *American Air Almanac*.

CHARLES H. SMILEY

EDITORIAL NOTE: According to information supplied to us by the publisher, tables identical with those of Martelli were published by a Mr. POUVREAU, in a French edition of 1885. This is evidently the following work in the Brown University Library: GEORGES POUVREAU, *Nouvelles Tables de Mer pour le calcul de la Hauteur de l'Heure et de l'Azimut*. Paris, Gauthier-Villars, 1885. xvi, 70 p. While it is true that the five tables of Pouvreau are based on the same formulae as those of Martelli, but entirely differently ordered, all of the Pouvreau tables are appreciably more extensive than those of Martelli, published up to 1885, or even later. The Tables 1, 4 and 5, to 4D in Martelli, are to 5D in Pouvreau, whose Table 1 is carried to 90°, instead of to 71° 34′. Martelli's Tables 2 and 3, giving values at interval 0″.1, are at interval 0″.01 in Pouvreau. Indeed it may well be that Pouvreau had no knowledge of Martelli's little volume of tables, which gave no formulae.

Martelli's tables have been discussed as follows in the U. S. Naval Institute, *Proc.*: J. P. JACKSON, "A short and simple method of finding the longitude at sea," v. 46, 1920, p. 739–742; H. V. HOPKINS, "Altitude by Martelli's tables," v. 59, 1933, p. 1171; H. J. RAY, "Altitude by Martelli's tables," v. 59, 1933, p. 1776–1777; H. V. Hopkins, "New Uses for Martelli's tables," v. 60, 1934, p. 651–654; HARRY LEYPOLDT, "Martelli's method," v. 61, 1935, p. 401. Mr. Hopkins, then a boatswain, but now a Lieutenant Commander in the U. S. Coast Guard, contributed also an article, "Martelli's tables," to *Nautical Mag.*, Glasgow, v. 138, 1937, p. 326–329. The 1932 edition of Martelli had only two pages of 1888 explanatory matter, but this was given in four languages, English, French, German and Italian; the same was true of the 1942 edition. But in 1943 the publishers issued a 12-page pamphlet (including the cover) *Supplement to Martelli's Navigational Tables*, and it is stated that "This supplement replaces the introduction and examples (pp. ii to viii) of the 1942 edition." It is entirely in English, incorporated methods and ideas from the articles of Mr. Hopkins, and included his computed extension of Table 1. This material was included in the 1944 and 1946 editions. The typography and paper of the present edition is better than in the earlier ones.

315[U].—P. V. H. WEEMS, (A) *Star Altitude Curves, . . . Latitudes 70° to 90° North*, third ed., and (B) *New Line of Position Tables*, Annapolis, Md., Weems System of Navigation. 1944, i–xiv, 701–716, 801–813 + [5] p. 21.5 × 35 cm. $7.00.

This volume contains star altitude curves for latitudes 70° to 90° North and tables which are suitable for use with any celestial body at any latitude. For use in the polar regions, some combination of this sort is necessary, since there are long intervals when only the sun, moon and one or two bright planets can be used due to the continuous presence of the sun in the sky, and there are other long intervals when the sun does not rise above the horizon.

Because this volume contains two essentially distinct methods, (A) and (B), under a single cover, they will be reviewed here as separate units.

(A) The horizontal scale of each of the charts for latitudes 70° to 80° North is local sidereal time which is given in hours and minutes at one-minute intervals at the top of the page and in degrees and minutes at 15′ intervals at the bottom of the page. The vertical scale is latitude in degrees and minutes at 5′ intervals. In the charts for 80° to 90° North latitude, the scale along the bottom of each page and along the right-hand side is local sidereal time, both hours and minutes, and degrees and minutes, with intervals of 2^m and 30′ respectively. Along the left and at the top are latitude scales, with 5′ intervals and with the pole (90° North latitude) in the upper left-hand corner.

On each page appear equal altitude curves for three stars of markedly different azimuths; those for each star are in a different color, red, green or black. The curves for altitudes measured in integral degrees are heavy and carry numerals indicating the altitude; somewhat lighter curves, one for each 10′ of altitude, are shown in between the heavy ones. The altitudes as shown on the charts are true altitudes increased by refraction (epoch of star positions Jan. 1, 1945); thus altitudes measured with a good bubble octant may be used without correction. Altitudes which have been measured with a marine sextant must be corrected for dip of the horizon before using the curves.

With the altitudes of two or three of the stars shown on the curves, taken within a few minutes of each other, one can enter this volume, sketch in the lines of position on the charts and determine a fix in a very few minutes. Neither a nautical almanac nor an air almanac is necessary if the user times his observations with a timepiece keeping Greenwich sidereal time.

One can easily estimate minutes of arc on all scales appearing on these charts; altitudes measured with a bubble octant aboard a plane are likely to be in error by several minutes. Furthermore most travel in the zone 70° to 90° North latitude will be by plane, so one may say that the star altitude curves in this volume are of adequate accuracy.

On the 16 pages covering latitudes 70° to 80° North, a total of eight stars appear: Vega on 12 pages, Deneb on 11, Arcturus 7, Alpheratz 5, Pollux 5, Capella 4, Dubhe 3 and Caph on one page only. On the 13 pages for 80° to 90° North latitude, only three stars are used, namely: Capella, Alkaid and Deneb. Five of the nine stars chosen are brilliant ones and favorites with navigators: Vega, Arcturus, Capella, Pollux and Deneb. The other four with their stellar magnitudes are Alpheratz 2.2, Caph 2.4, Dubhe 2.0, and Alkaid 1.9; they will be somewhat harder to observe. Fortunately on no page does more than one of these faint stars appear.

Probably the one criticism which is raised most often against the star altitude curves is that the method is graphical. Many persons of strong mathematical background are allergic to graphical methods because of their general lack of satisfactory accuracy. However, as pointed out above, this method, though graphical, provides accuracy somewhat greater than the observations will require. Another criticism to be made of the curves is that there is very little overlap (one degree in the 70°–80° North latitude section) between the charts given on successive pages, or at top and bottom between successive latitude sections (30′). In this connection, it may be pointed out that page III is headed "STAR

ALTITUDE CURVES, Latitude 70° to 80° North," and yet just above DIRECTIONS FOR USE appears the statement, "This edition covers latitudes 0 to 70°30′."

A weakness of the curves is that one must observe at least two of the three stars given for the particular time of the observations and within the time interval for which they appear. For example, if the local sidereal time is 9 hours 10 minutes instead of an estimated 9 hours, and one has observed Caph at latitude 75° North, one will find that the fix will lie outside the chart. The advantage to be gained by the use of a sidereal timepiece will usually be lost, since such instruments are not common.

(B) These tables represent an improved version of the author's *Line of Position Book*, RMT 298; their three principal advantages are improved legibility, increased accuracy, and better arrangement. In particular, all of the data in Table A concerning one latitude are now found on a single page. Since latitude seldom changes as rapidly as local hour angle, even with the breath-taking speeds of today, this represents a real improvement over the arrangement to be found in the *Line of Position Book*. The numerical values of the tabulated quantities appear to be identical with the corresponding values in *Hughes' Tables for Sea and Air Navigation* by L. J. COMRIE (RMT 115) and are therefore the most accurate values of this sort in print today.

Table A is a double-entry table with horizontal argument latitude 0 to 89° by integral degrees, five values to a page, and with vertical argument local hour angle, 0(1°)90° on the left side of the page, and 90°(1°)180° on the right side of the page. For local hour angle 0 to 90°, the tabulated quantities are K, A, and Z_1; for local hour angles 90° to 180°, they are 180°-K, A, and Z_1. K and A are as defined in the *Line of Position Book*. K is given to the nearest 0′.1 and values of A less than 665 are given to one decimal, other values of A to the nearest integer. Z_1 is the angle at the zenith subtended by the arc of the hour circle of the celestial object between the elevated pole and the foot of the perpendicular from the zenith on this hour circle; it is given to the nearest 0′.1. Table B is essentially the same as Table B in the *Line of Position Book*, and Weems offers the same formula for determining the altitude, h, as in the older book.

The author names four methods of determining the azimuth, listing them in order of difficulty, the simplest first. They are:

(a) By direct observation with a pelorus or other instrument.
(b) By the use of Rust's modified azimuth diagram, a copy of which is included (see below).
(c) By the use of Table A in obtaining Z_2. One can either replace L, t, and A by h, Z_2, and $B(K \sim d)$ respectively, counting h and B as known and determining Z_2; or replace L, A, Z_1 by h, A, Z_2 respectively, counting h and A as known and determining Z_2. The rule for combining Z_1 and Z_2 to form Z is a trifle complicated.
(d) By the use of the formula $\csc Z = \csc t \sec d / \sec h$ and Table B.

The azimuth diagram of RUST, originally appearing in his *Ex-meridian Altitude, Azimuth and Star-Finding Tables* (N. Y., Wiley, 1908), plate VI, is reproduced here as a single unit on a scale approximately 1.4 times as large. Otherwise the diagram and auxiliary diagram are as described in RMT 298.

Also included in the volume are tables for the correction of observed altitudes for dip of horizon, refraction, and parallax, all well designed for convenient use. Of especial interest are two pages of correction tables for altitudes between 0 and 6°, with special "sub-correction tables" to take care of variations in height above sea level, temperature and pressure. The author warns the user, however, "that refractions at these low altitudes are on rare occasions subject to large and unpredictable variations."

Two pages are devoted to explaining the methods which may be used in adjusting for run between sights or for time elapsed, in the polar regions, including the use of the Greenwich meridian as a basis for describing directions, the so-called G system.

Illyne's Star Chart is presented in a reduced form, opposite a page of material designed to help the navigator identify the principal stars and constellations as they will be seen from the north polar regions of the earth.

CHARLES H. SMILEY

MATHEMATICAL TABLES—ERRATA

References have been made to Errata in RMT **289** (Theodorsen), **304** (Editorial N.), **311** (Hayashi, Milne-Thomson), **312** (Benest & Timberlake), **313** (Hoehne), **314** (Martelli).

84. BAASMTC, *Mathematical Tables*, volume 1, London, 1931. This v. has long been out of print. It has been announced that the following errors are corrected in the new edition. Compare *MTAC*, v. 1, p. 323, and RMT **303**.

TABLE II (p. 5, 7)

cos 26.1: *for* .56756..., *read* .56755... sin 47.6: *for* .46832..., *read* .45832...

TABLE VII (p. 32)

Ei(5.3): *for* ...031, *read* ...030 Ei(6.7): *for* ...344, *read* ...345
Ei(5.6): *for* ...598, *read* ...597 Ei(8.0): *for* ...954, *read* ...953
Ei(5.9): *for* ...015, *read* ...014 Ei(9.8): *for* ... 35, *read* ... 34
$-\mathrm{Ei}(-7.3)$: *for* ...4446, *read* ...4445

Consequent changes

Corrected end figures of actual differences:

x	5.2	5.3	5.4	5.5	5.6	5.7	5.8	5.9	6.0
$\delta^3\mathrm{Ei}(x)$	692	571	295	327	155	484	468	921	585

x	6.6	6.7	6.8	7.9	8.0	8.1	9.7	9.8	9.9
$\delta^3\mathrm{Ei}(x)$	086	208	940	815	887	199	79	20	30

x	7.8	7.9	8.0	8.1	8.2
$\delta^4\mathrm{Ei}(x)$	619	787	240	952	073

x	7.0	7.7	7.8	7.9	8.0	8.1	8.2	8.3
$\delta^4\mathrm{Ei}(x)$	815	855	480	285	259	409	808	391

x	7.2	7.3	7.4
$\delta^2\{-\mathrm{Ei}(-x)\}$	8075	4950	9472

x	7.2	7.3	7.4	7.5
$\delta^4\{-\mathrm{Ei}(-x)\}$	9630	7647	6709	4290

x	7.2	7.3	7.4	7.5	7.6
$\delta^5\{-\mathrm{Ei}(-x)\}$	3852	1045	8519	6346	4426

Corrected values of modified differences:

$\delta^4\mathrm{Ei}(5.6)$ *should read* 299 994 $\delta^4\{-\mathrm{Ei}(-7.1)\}$ *should read* 180 986

Corrected end figures of modified differences:

x	5.1	5.2	5.3	5.4	5.5	5.6	5.7	5.8	5.9	6.0	6.1
$\delta^4\mathrm{Ei}(x)$	005	898	459	889	337	994	104	863	547	433	834

x	6.5	6.6	6.7	6.8	6.9	9.6	9.7	9.8	9.9	10.0
$\delta^4\mathrm{Ei}(x)$	739	438	227	682	467	75	29	01	81	25

TABLE VIII (p. 35)

Si(8.8): *for* ...21860, *read* ...21861 Ci(5.5): *for* ...29475, *read* ...29476

Consequent changes

Corrected end figures of actual differences:

x	8.7	8.8	8.9
$\delta^3\mathrm{Si}(x)$	303	361	485

x	5.4	5.5	5.6
$\delta^3\mathrm{Ci}(x)$	462	196	704

Corrected end figures of modified differences:

x	8.6	8.7	8.8	8.9	9.0	x	5.3	5.4	5.5	5.6	5.7
$\delta^4 \mathrm{Si}(x)$	542	853	146	410	663	$\delta^4 \mathrm{Ci}(x)$	505	835	200	685	428

TABLE IX (p. 40)

0.27!: *for* ...4465, *read* ...4466

With this change the table, originally described as of slightly less than 12-figure accuracy, becomes accurate to 12 figures throughout.

Consequent changes

Corrected end figures of actual differences:

x	0.26	0.27	0.28
$\delta^2(x!)$	899	391	728

Corrected end figures of modified differences:

x	0.24	0.25	0.26	0.27	0.28	0.29
$\delta^4(x!)$	622	531	605	819	153	631

85. A. FLETCHER, J. C. P. MILLER and L. ROSENHEAD. *An Index of Mathematical Tables*, [FMR *Index*], London, Scientific Computing Service, 1946.

The gratifying review of this book, RMT **233**, *MTAC* v. 2, p. 13f, appeared just before the final printing of the work and the opportunity was taken to make three alterations: those indicated in the review (i) to the heading of Art. 5.745 on p. 105, (ii) to Hutton 1775 on p. 404 and (iii) to Kulik 1860 on p. 409.

Three errors have come to our notice:

p. 300. Art. 20.67. Insert ∞ for the upper limit to the integral for $C_1(u)$. This was correct in the proof that was passed for press, and may not be missing from all copies.

p. 381. British Association Mathematical Tables. The statement that the first five volumes are out of print is incorrect. At the time the passage was written, volumes I and II were out of print. Since then, there has been a second edition of volume I, while volume VI has now run out of print. Volumes III, IV and V were very difficult to obtain, mainly owing to war-time conditions, but they were not, in fact, out of print.

p. 387. Under Crelle, *for* 1864, *read* 1857.

J. C. P. MILLER

EDITORIAL NOTE: FMR *Index* was published in England in mid-April and sheets were then sent to McGraw-Hill & Co., New York, for binding, and distribution in the Western Hemisphere. In our review, *MTAC*, v. 2, p. 18, line 16–18, it is not brought out that the *Index* referred to "editions" in 1890, both by Dickstein; one of them, however, was but a small portion of the original of HOËNE-WROŃSKI.

86. A. M. LEGENDRE, *Tafeln der Elliptischen Normalintegrale erster und zweiter Gattung*, hrsg. von FRITZ EMDE, Stuttgart, 1931. This is a facsimile reproduction of Tables VIII and IX, to 10D in Legendre's *Exercices de Calcul Intégral*, Paris, v. 3, 1816, p. 338–416.

In *Z. angew. Math. Mech.*, v. 21, Aug. 1941, p. 254, Emde reports the following six errors in $F(\phi, \theta)$, found by GUSTAV WITT:

Page	ϕ	θ	*For*	*Read*
350(15)	20°	9°	0.34293	0.34923
364(29)	*90°	23°	1.63631	1.63651
368(33)	*85°	27°	1.56840	1.56480
380(45)	*80°	44°	1.58906	1.59806
396(61)	*79°	64°	1.84693	1.84793
416(81)	49°	87°	0.98238	0.98328

Those errors, as well as others below, marked with a star (*) were already reported in N. Samoĭlova-Îakhontova, *Tablitsy Ellipticheskikh Integralov*, Moscow and Leningrad, 1935, p. 6, and in *Scripta Math.*, v. 3, 1935, p. 365. All errors both here and below were listed by Heuman, April, 1941 (see *MTAC*, v. 1, p. 187; an additional errata sheet by Heuman, dated June 1941, was printed at Stockholm). Seventeen other errors in $F(\phi, \theta)$ are as follows:

Page	ϕ	θ	For	Read
346(11)	42°	2°	0.73311 099	0.73311 009
350(15)	5°	7°	0.08729 8	0.08726 8
358(23)	32°	20°	0.56174 72	0.56174 52
364(29)	*88°	21°	1.58584	1.58784
366(31)	*35°	30°	0.62003	0.62203
	42°	27°	0.74754	0.74574
368(33)	59°	27°	1.06251 39	1.06251 29
384(49)	51°	46°	0.95158 7	0.95157 7
392(57) and 396(61)	*60°	60°	1.21253 6	1.21259 6
392(57)	*65°	59°	1.34195 7	1.34196 7
	90°	57°	2.08035 816	2.08035 806
404(69)	61°	71°	1.29179	1.29719
410(75)	18°	84°	0.31936 7	0.31939 7
412(77)	*74°	83°	1.92525	1.92515
416(81)	*86°	86°	1.17204 1744	1.17024 9982[1]
	87°	87°	3.45644 5172	3.45667 6096[1]

Samoĭlova-Îakhontova and Heuman used the 1826 edition of Legendre's tables in his *Traité des Fonctions Elliptiques*, v. 2. There are here at least five errors which do not occur in the 1816 edition, namely in the values for: $\phi = 1°$, $\theta = 14°$; $\phi = 37°$, $\theta = 75°$; $\phi = 4°$, $\theta = 88°$; $\phi = 31°$, $\theta = 90°$. Also in $E(\phi, \theta)$, $\phi = 5°$, $\theta = 10°$. Two of these were noted by K. Bohlin (1900). FMR *Index* (RMT 233) states, p. 316, "A few of the errors in Legendre 1816 are corrected in Legendre 1826." In the present check we have found that all the errors of Legendre 1816 are in Legendre 1826 and also five new ones, not in Legendre 1816.

There are the following 7 errata in $E(\phi, \theta)$:

Page	ϕ	θ	For	Read
347(12)	51°	4°	0.88952	0.88962
351(16)	*90°	8°	1.56296	1.56316
363(28)	78°	22°	1.31072	1.31972
377(42)	27°	42°	0.46376	0.46366
379(44) and 383(48)	73°	45°	1.13785 83	1.13785 43
405(70)	32°	79°	0.53101 73	0.53101 13.

Since the K. Pearson facsimile was of Legendre 1826 (Pearson has 1825!, Cambridge, 1934), it is clear from what we have indicated above that there are in it at least 35 serious errors in E and F alone. A facsimile of Legendre 1816 tables is also in L. Potin, *Formules et Tables Numériques*, Paris, 1925. Hence the 30 errors indicated above for the Legendre-Emde tables occur also in these tables.

R. C. A.

[1] These numbers are from the errata list of Heuman who states "The last decimals are uncertain."

87. T. J. STIELTJES, "Table des valeurs des sommes $S_k = \sum_1^\infty n^{-k}$," *Acta Mathem.*, v. 10, 1887, p. 299–302. J. W. L. GLAISHER, "Tables of $1 \pm 2^{-n} + 3^{-n} \pm 4^{-n} + \cdots$ and $1 + 3^{-n} + 5^{-n} + 7^{-n} + \cdots$ to 32 places of decimals," *Quart. J. Math.*, v. 45, 1914, p. 148–150. See also UMT 46.

The Stieltjes table of $S_n = 1 + 2^{-n} + 3^{-n} + \cdots$ is for $n = [2(1)70; 32D]$. Glaisher gave a table of S_n for $n = [2(1)107; 32D]$, but he copied from the table of Stieltjes the values for $n = 2(1)33$.

In the *Anhang* (p. 90) by PETERS & STEIN, of J. T. PETERS, *Zehnstellige Logarithmentafel*, v. 1, Berlin, 1922, there is a table of $S'_n = S_n - 1$ for $n = 2(1)100$. On comparing this table with that of Stieltjes, I found that there were 23 differences in the end-figures of the first 31 values of S_n. Upon checking the values I found that Peters & Stein were correct in every case except one, $n = 25$, so that there were the following 23 errors in the part of the table published by Stieltjes, and later copied by Glaisher. Apart from the 15 cases of unit errors, there were 8 cases of errors of two units in the final digit; S_7, S_{10}, S_{16}, S_{20}, S_{21} were each 2 units in excess of the correct values, but S_{12}, S_{17}, and S_{28} were each 2 units in defect. As to the unit errors there were errors of excess in S_2, S_4, S_6, S_{13}, S_{15}, S_{23}, S_{27}, S_{32}; and errors of defect in S_9, S_{11}, S_{18}, S_{22}, S_{24}, S_{26}, S_{29}. The error noted in S_3 was also checked by D. H. LEHMER, to 100D, *Scripta Mathematica*, v. 4, 1936, p. 293, and by J. W. BRADSHAW, *Amer. Math. Mo.*, v. 51, 1944, p. 390. My last seven calculated figures of S_n to 37D (a) in each of the 23 cases where Stieltjes and Peters & Stein did not agree, (b) in the other 9 values of n, not recalculated by Glaisher, up to and including $n = 33$, are as follows:

S_2	0251892	S_{10}	3190170	S_{18}	6219397	S_{26}	5066307
S_3	4499908	S_{11}	4699365	S_{19}	7951014	S_{27}	0041706
S_4	1679028	S_{12}	7396710	S_{20}	6834493	S_{28}	2040184
S_5	0341681	S_{13}	3573957	S_{21}	6043727	S_{29}	9099454
S_6	9205279	S_{14}	1353337	S_{22}	1867530	S_{30}	7647350
S_7	7967596	S_{15}	6450626	S_{23}	7188823	S_{31}	9233251
S_8	6524653	S_{16}	6367220	S_{24}	2079358	S_{32}	1455976
S_9	4120605	S_{17}	5630292	S_{25}	7050694	S_{33}	2973835

All the data were simultaneously checked by means of the equation

$$\sum_{n=2}^{33} S'_n = 1 - \sum_{n=2}^{\infty} (n-1)^{-1} n^{-33}.$$

The respective members of this equation differed by 3.5×10^{-37} when my 37D approximations to S_n were used in evaluating the left side. This discrepancy is due to the rounding of the data to 37 decimal places.

Glaisher corrected 10 unit errors in final digits of Stieltjes S_n, $n = 39, 42, 43, 46, 47, 56, 57, 65$ in excess; $n = 61, 67$ in defect. The Stieltjes table did not profess to be correct in the final decimal place. The Glaisher table was exactly reprinted in H. T. DAVIS, *Tables of the Higher Mathematical Functions*, v. 2, Bloomington, Indiana, 1935, p. 244, 218. Hence there are 23 errors, indicated above, in each of the Glaisher and Davis tables. In addition it may be mentioned that I have computed S_n more extensively, $n = 2(2)20$, to 52D, and $n = 3, 5$, to 42D.

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UNPUBLISHED MATHEMATICAL TABLES

For other unpublished tabular numbers see MTE **87** (Lehmer and Wrench).

46[B, J].—ENZO CAMBI, Tables of $S_n = 1 + 2^{-n} + 3^{-n} + \cdots$, and of $(1 - x^2)^{\frac{1}{2}}$, mss. in possession of, and prepared by the author, Via Giovanni Antonelli 3, Rome, Italy.

I have computed S_n, for n = [3(2)71; 42 to 60D]; also a table of $(1 - x^2)^{\frac{1}{2}}$, x = [0(.001)1; 16D].

E. CAMBI

EDITORIAL NOTE: Compare MTE **87**, and *MTAC*, v. 1, p. 456–457. Of the function $(1 - x^2)^{\frac{1}{2}} = P_{1,1}(x)$, there are tables in A. H. H. TALLQUIST Finska Wetenskaps-Societeten, *Acta*, v. 33, 1906, p. 9, x = [0(.01)1; 10D]; in T. L. KELLEY, *The Kelley Statistical Tables*, New York, 1938, p. 14f, x = [0(.0001)1; 8D]; and in J. R. MINER, *Tables of* $\sqrt{1 - r^2}$ *and* $1 - r^2$ *for use in partial correlation and in Trigonometry*, Baltimore, Md., 1922, $x = r$ = [0(.0001)1; 6D].

47[E].—K. S. JOHNSON & P. H. RICHARDSON, *Tables of Useful Exponential and Hyperbolic Functions*. 16 p. 22 × 25.5 cm. Ms. property of the Bell Telephone Laboratories, 463 West St., New York City.

In RMT **243**, reference was made to the first published mathematical tables, 1945, with arguments decibels (N), and the corresponding number of nepers (x). There, for N = [.05, .1(.1)2(.2)4(.5)20(1)50(5)100; 5–6S], are given values of x, e^x, e^{-x}, e^{2x}, sinh x, cosh x, tanh x, tanh $(\frac{1}{2}x)$. Johnson & Richardson's tables, calculated in October, 1929 (with revisions at the beginning and end in August, 1935) are of x, e^x, e^{-x}, sinh x, cosh x, and tanh x, for N = [.001(.001).01(.005)1(.1)30(.2)55; 5–7S]. Near the beginning of the table are some entries of more than 7S.

For large values of N the corresponding values of e^x or e^{-x} may be obtained by moving the decimal point to the right or left one point for each 20 N. For example, corresponding to N = 75 (from N = 55), e^x = 5623.416 and e^{-x} = .00017783.

Sinh x and cosh x may then be obtained from the rigorous relations

$$\sinh x = \tfrac{1}{2}e^x - \tfrac{1}{2}e^{-x}, \quad \cosh x = \tfrac{1}{2}e^x + \tfrac{1}{2}e^{-x};$$

or from the approximate relationship

$$\sinh x \doteq \cosh x \doteq \tfrac{1}{2}e^x, \quad \tanh x = \sinh x/\cosh x \doteq 1 - 2e^{-2x}.$$

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48[F].—KULIK's *Magnus Canon Divisorum* . . ., 8 ms. volumes deposited in the Library of the Academy of Sciences, Vienna, after Kulik's death in February 1863; see Akad. d. Wissen., Vienna, *Sitzungsb.*, v. 53, 1866, p. 460–462, and POGGENDORFF, *Biog.-Literar. Handwörterbuch*, v. 3, 1897, p. 757. Compare MTAC, v. 2, p. 30, 59–60.

EDITORIAL NOTE: This great factor table of numbers to 100,330,200 (except for multiples of 2,3,5) has the following title: *Magnus Canon Divisorum pro omnibus numeris per 2, 3, et 5 non divisibilibus, et numerorum primorum interjacentium ad millies centena millia accuratius ad* 100 330 201 *usque*. Details of the notation and of other matters connected with the table were set forth by D. N. LEHMER in his 1. *Factor Table for the First Ten Millions* . . ., Washington, 1909, cols. IX, X, XIII, XIV, and 2. *List of Prime Numbers from* 1 *to* 10,006,721, Washington, 1914, cols. XI, XII. Lehmer mentions that volume 2 of Kulik's table was already missing 35 years ago. Mainly with information furnished by Lehmer, S. A. JOFFE figured out the exact contents of each of the 8 volumes. On submitting this survey to the Librarian of the Academy of Sciences, W. OBERHUMMER, we learned from him,

by a letter dated 21 February 1946, that the analysis was accurate in every particular; that volume 2 was still missing; and that the other 7 volumes are intact in the Academy's Library. Mr. Joffe's report now follows.

As indicated in *MTAC*, v. 2, p. 30, D. N. Lehmer published information regarding the number of pages in each of the extant volumes, and the corresponding range of consecutive integers covered in each volume. A study of Lehmer's results, which are incorrect in several instances, enabled the writer to correct Lehmer's errors and to present the consecutive results in tabular form.

When Kulik commenced work on his factor table, there were no other factor tables but those of J. Ch. Burckhardt, *Table des Diviseurs*, v. 1, 1817, v. 2, 1814, v. 3, 1816, and Kulik began where Burckhardt left off. However, the last integer in Burckhardt's v. 3 is 3,036,000, but for some reason not yet ascertained the opening number in Kulik's v. 1 is 3,033,001.

Disregarding Kulik's missing v. 2, we find that in all the following volumes the tables are written on both sides of the paper. On the assumption that the methodical and economical Kulik would cover each page in full, without blank spaces, and that therefore the right and left sides of each volume must contain the same number of pages, (and also of consecutive integers), it was established that each page covered 23,100 consecutive integers. Incidentally, from this we see that a million consecutive integers would require, roughly 43½ p., as stated in *MTAC*, v. 2, p. 30.

The following table gives in detail the contents of each v., both as to the number of pages and the number of consecutive integers.

Volume r=right l=left	Number of p. in vol.	Serial number of p. in vol.	Number of Consec. Integers in vol.	Cumulative Consecutive Integers in set of 8 vol.
I	416	1 –416	9,609,600	3,033,001 – 12,642,600
II	442	417 –858	10,210,200	12,642,601 – 22,852,800
III r	277	859[1]–1135	6,398,700	22,852,801 – 29,251,500
IV r	276	1136 –1411	6,375,600	29,251,501 – 35,627,100[2]
V r	276	1412 –1687	6,375,600	35,627,101 – 42,002,700
VI r	276	1688 –1963	6,375,600	42,002,701 – 48,378,300
III l	277	1964 –2240	6,398,700	48,378,301 – 54,777,000
IV l	276	2241 –2516	6,375,600	54,777,001 – 61,152,600
V l	276	2517 –2792	6,375,600	61,152,601 – 67,528,200
VI l	276	2793 –3068	6,375,600	67,528,201 – 73,903,800
VII r	285	3069 –3353	6,583,500	73,903,801[3]– 80,487,300
VIII r	287	3354 –3640	6,629,700	80,487,301 – 87,117,000
VII l	285	3641 –3925	6,583,500	87,117,001 – 93,700,500[4]
VIII l	287	3926 –4212	6,629,700	93,700,501 –100,330,200

[1] Lehmer has 828.
[2] Lehmer has 35,626,799(800).
[3] Lehmer has 79,903,801.
[4] Lehmer has 93,709,499(500).

S. A. Joffe

49[L].—NYMTP, *Tables of Intensity Functions*. Tables prepared by and in possession of the NYMTP.

A. Several years ago the NYMTP, upon the recommendation of V. K. LaMer, professor of chemistry at Columbia University, then a member of Division 10 of the National Defense Research Council (of the Office of Scientific Research and Development), computed extensive tables of Intensity Functions for both real and complex values of the index of refraction. The pertinent tabular material relating to real indices of refraction has since been incorporated in a joint report by LaMer & Sinclair, entitled *Progress Report on the Verification of Mie's Theory—Calculations and Measurements of Light Scattering by Dielectric Spherical Particles*, OSRD Report 1851, September 29, 1943. A limited number of mimeographed copies of this report have been made available to the OSRD for distribution to those working on problems related to scattering of light by spherical particles. A detailed

description of the complete manuscript on Intensity Functions, prepared by the NYMTP for both real and complex indices of refraction, is given below. The physical significance of the tabulated functions is discussed in B.

(1) *First Part*

Definitions:

$$i_1 = \left| \sum_{n=1}^{\infty} \{A_n \pi_n + P_n[x\pi_n - (1-x^2)\pi_n']\} \right|^2$$

$$i_2 = \left| \sum_{n=1}^{\infty} \{A_n[x\pi_n - (1-x^2)\pi_n'] + P_n \pi_n\} \right|^2$$

$$\tfrac{1}{2}\alpha^2 K(m;\alpha) = \tfrac{1}{2}\int_0^{\pi} (i_1 + i_2) \sin\gamma d\gamma = \sum_{n=1}^{\infty} \frac{R^2(A_n) + I^2(A_n) + R^2(P_n) + I^2(P_n)}{(2n+1)/n^2(n+1)^2}$$

$$A_n = a_n/n(n+1); \qquad P_n = p_n/n(n+1); \qquad x = \cos\gamma,$$

$$a_n = (-1)^{n+1}(2n+1)\frac{S_n'(\beta)S_n(\alpha) - mS_n'(\alpha)S_n(\beta)}{S_n'(\beta)\varphi_n(\alpha) - m\varphi_n'(\alpha)S_n(\beta)},$$

$$p_n = (-1)^{n+1}(2n+1)\frac{mS_n(\alpha)S_n'(\beta) - S_n(\beta)S_n'(\alpha)}{m\varphi_n(\alpha)S_n'(\beta) - S_n(\beta)\varphi_n'(\alpha)};$$

and

$$S_n(x) = (\tfrac{1}{2}\pi x)^{\frac{1}{2}} J_{n+\frac{1}{2}}(x); \qquad C_n(x) = (-1)^n(\tfrac{1}{2}\pi x)^{\frac{1}{2}} J_{-n-\frac{1}{2}}(x);$$

$$\pi_n = \pi_n(x) = \frac{\partial P_n(x)}{\partial x}; \qquad \pi_n' = \pi_n'(x) = \frac{\partial^2 P_n(x)}{\partial x^2};$$

$$\varphi_n(x) = S_n(x) + iC_n(x); \qquad \beta = m\alpha; \qquad \alpha = 2\pi r/\lambda.$$

$R(A_n)$, $I(A_n)$, $R(P_n)$ and $I(P_n)$ stand for the real and imaginary parts of A_n and P_n respectively.

$J_{n+\frac{1}{2}}(x)$ and $J_{-n-\frac{1}{2}}(x)$ are Bessel Functions of half-integral order. $P_n(x)$ is the Legendre Polynomial.

The tabulated material consists of values of i_1, i_2, $R(A_n)$, $I(A_n)$, $R(P_n)$ and $I(P_n)$ (the number of decimal places and significant figures varying), for $m = 1.33, 1.44, 1.55, 2.0$; $\alpha = .5, .6, 1, 1.2, 1.5, 1.8, 2, 2.4, 2.5, 3, 3.6, 4, 4.8, 5, 6$; $\gamma = 0(10°)180°$.

The integral $\frac{1}{2}\int_0^{\pi} (i_1 + i_2) \sin\gamma d\gamma$ is given to 3S for the above values of m and α.

(2) *Second Part*

$$K(m;\alpha) = \frac{2}{\alpha^2} \text{Real}\left\{\sum_{n=1}^{\infty} (-1)^n(2n+1)(C_n^1 + C_n^2)\right\},$$

where

$$C_n^1 = -ia_n/(2n+1), \qquad C_n^2 = ip_n/(2n+1),$$

a_n and p_n having the same meaning as in (1).

The tabulated material consists of

$$K(m;\alpha), \quad (-1)^n(2n+1)\,\text{Real}\,(C_n^1) \quad \text{and} \quad (-1)^n(2n+1)\,\text{Imag}\,(C_n^1)$$

for $m = 1.5$; $\alpha = .5, .6, 1(.5)3, 1.2(1.2)12, 1.8, 3.2, 3.4, 3.8, 4(.5)5, 5.3, 7(1)10$; while n varies from a maximum of 1 for $\alpha = .5$ to a maximum of 16 for $\alpha = 12$.

$K(m;\alpha)$ is given everywhere to 4S while the other functions are given to a constant number of D (predominantly 4).

(3) *Third Part*

$$F(m;\alpha) = K(m;\alpha) + iL(m;\alpha)$$

where $F(m;\alpha)$ is the complex function defined under (2) i.e. $K(m;\alpha) = \text{Real}\,\{F(m;\alpha)\}$.

If in above expression m is replaced by $z = m - imk$ then

$$F(z;\alpha) = F(m - imk;\alpha) = K(m, -mk;\alpha) + iL(m, -mk;\alpha).$$

The function desired was $K(m, -mk;\alpha)$ for $m = 1.44, 1.55$, the same values of α listed under (1) in addition to $\alpha = 3.2, 4.5$, and $k \leqq 10^{-1}$. Instead of tabular values of $K(m, -mk;\alpha)$ this section contains

(a) Approximation polynomials for $K(m;\alpha)$, $L(m;\alpha)$, $K(1.5, -1.5k;\alpha)$ and $K(m, -mk;\alpha)$

(b) Tables of $K(m;\alpha)$ and $L(m;\alpha)$ for the above values of α and m = [1.44(.01)1.55; 4D or 5D]

A. N. LOWAN

B. These Intensity Functions give the angular distribution of intensity and the total light scattered by a small, spherical particle (such as a fog droplet) as a function of the parameter, $\alpha = 2\pi r/\lambda$, when the particle radius, r, is roughly equal to the wavelength, λ, of the incident light. The angular distribution and total light scattered by a suspension of particles of uniform size can be used to determine the particle size and concentration as described below.

In Part 1, the angular distribution functions i_1 and i_2 are proportional to the intensities of the two incoherent, plane polarized components scattered by a transparent particle illuminated with natural light. i_1 is the component whose electric vector is perpendicular to the plane of observation (the plane containing the direction of observation and the direction of propagation of the incident light), and i_2 is polarized perpendicular to i_1. When the particle is illuminated by two (2) units of energy per unit cross-sectional area, the actual intensities scattered per unit solid angle in the direction γ are $\lambda^2 i_1/(4\pi^2)$ and $\lambda^2 i_2/(4\pi^2)$. γ is the angle between the direction of propagation of the scattered light and the reversed direction of propagation of the incident light.

The polarization, i.e., the relative values of i_1 and i_2 (at a fixed angle γ), measured with a polarization photometer, provide a measure of particle size from .05 to .2 micron radius. Also, the angle of maximum polarization may be used.

When the particle is illuminated by polarized light of unit intensity, $\lambda^2 i_1/(4\pi^2)$ is the intensity scattered in a direction perpendicular to the incident electric vector, and $\lambda^2 i_2/(4\pi^2)$ is the intensity scattered in the plane containing the incident electric vector and its direction of propagation. The light scattered in any other directions by a particle illuminated by plane polarized light is elliptically polarized and i_1 and i_2 do *not* give its components.

Pairs of values of α, whose ratio is 1.2, were chosen for each value of the refractive index m. 1.2 is the wavelength ratio of red light ($\lambda = .629\mu$) to that of green light ($\lambda = .524\mu$), the most distinctive colors observed. Consequently, each pair of α values yields the intensities scattered at these two wavelengths (or any other pair of wavelengths bearing the same ratio) by a particle of a given radius.

The relative intensity of red to that of green in either i_1 or i_2 alone is a function of the angle γ. The number and angular position of the maxima of this function provide a measure of particle size from about .1 to 1.2 micron radius. The particle radius in microns is given roughly by dividing the number of maxima in i_1 by 10.

The integral or the sum when multiplied by $\lambda^2/(2\pi)$ gives the total light scattered by a particle when illuminated at unit intensity.

In Part 2, the total scattering coefficient, K, defined as the total energy scattered per second per unit cross-sectional area of particle (illuminated at unit intensity), is given for a transparent material of refractive index 1.5, over a greater range of α values, chosen as described in Part 1. (K can be obtained for the refractive indices in Part 1 by multiplying the integral or sum in Part 1 by $2/\alpha^2$).

In Part 3, the total scattering (and absorbing) coefficient, K, may be obtained for absorbing materials of extinction coefficient, k, ($e^{-4\pi k}$ is the fraction of light absorbed in travelling a distance equal to one wavelength through the bulk material) varying from zero

to .1, and of real refractive index varying from 1.44 to 1.55. For the higher values of the extinction coefficient, the total scattering coefficient may be in error by 1%. When the extinction coefficient is zero, approximate values of the scattering coefficient will be obtained for transparent materials of refractive index as low as 1.33 and as high as 1.65 by substituting into the polynominal expression for $K(m;\alpha)$.

The number concentration, n, or the particle size, can be obtained when either is known by measuring the transmission at a known wavelength; or both can be obtained by measuring the transmission at two or more wavelengths. The transmission, $T = e^{-K\pi r^2 nl}$, where $K\pi r^2 n$ is the "absorption" coefficient as usually defined, i.e., $e^{-K\pi r^2 nl}$ is the fraction of light scattered by transparent particles or scattered and absorbed by absorbing particles per unit distance in the suspension.

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MECHANICAL AIDS TO COMPUTATION

See also the two introductory articles of this issue, as well as RMT **305**.

22. DE FOREST ON ELECTRICAL COMPUTERS.—To the *Yale Scientific Monthly*, published by the Senior class of the Sheffield Scientific School, Yale University, LEE DE FOREST (1873–) contributed an article ("A Wheatstone bridge for solving numerical equations," v. 3, 1897, p. 200–206),[1] explaining an idea of his for the electrical solution of equations. After describing a scheme in which the operator achieved his solution by adjusting the sliders on resistance potentiometers so as to balance a Wheatstone bridge, he pointed to the possibility of an "automatic balancer." He believed that "a relay type galvanometer driving or reversing some electrically governed mechanism might be devised, which would keep this length shifting until balance was attained."

If not the first, this is at least a very early description of the self-centering servomechanism as a computing device, the basic unit of some of the most important military computers of the recent war, including the electrical gun director for the control of antiaircraft fire.

DeForest may be pardoned for his qualifying remark—"it could not be very accurately done"—since, presumably, he did not yet know that he was going to invent the three-element thermionic valve, which did so much to make high precision possible.

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[1] EDITORIAL NOTE: This machine is one which escaped the attention of J. S. FRAME in his survey article "Machines for solving algebraic equations," *MTAC*, v. 1, p. 337–353. He did, however, deal with the machines of G. B. GRANT, C. V. BOYS, L. L. C. LALANNE referred to by DeForest in his introductory paragraphs. DeForest adds the new remark, that A. W. PHILLIPS (1844–1915) of Yale University was in 1879 an independent inventor of Lalanne's machine. Phillips was dean of the Graduate School 1895–1911, and the joint author of a number of texts in elementary mathematics.

NOTES

56. APPROXIMATIONS TO π.—In R. So. London, *Trans.*, 1841, p. 281–283, WILLIAM RUTHERFORD (1798?–1871) gave a value of π to 208D, which was correct to 152D, and derived from the Euler formula (1764)

$$\frac{\pi}{4} = 4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{70} + \tan^{-1}\frac{1}{99}. \tag{1}$$

With a formula, apparently due to L. K. SCHULZ VON STRASSNITZKY (1803–1852),

$$\frac{\pi}{4} = \tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{8}, \tag{2}$$

ZACHARIAS DASE (1824–1861) computed correctly to 200D, *J. f. d. reine u. angew. Math.*, v. 27, 1844, p. 198.

Three years later a value computed by THOMAS CLAUSEN (1801–1885), and correct to 248D, was published in *Astron. Nach.*, v. 25, 1847, col. 207f. Clausen used the formula of MACHIN (1706)

$$\frac{\pi}{4} = 4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{239}, \tag{3}$$

and a formula of EULER (1779) *Opera Omnia*, s. 1, v. 16_2, 1935, p. 26; also used by VEGA (1794)

$$\frac{\pi}{4} = 2\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{7}. \tag{4}$$

In *Phil. Mag.*, s. 4, v. 5, 1853, p. 214, Rutherford indicated the value of π calculated to 440D by means of Machin's formula (3). Using the same formula, WILLIAM SHANKS (1812–1882) gave full details of his work in his *Contributions to Mathematics comprising chiefly the Rectification of the Circle to 607 Places of Decimals*, London, 1853. These calculations were correct to only 459D. Rutherford had, however, published (p. 214) Shanks' value to 530D, which is correct to 500D, at least, as may be checked by the independent computation (climaxing two previous publications) of RICHTER (d. 1854) in *Archiv Math. Phys.*, v. 25, 1855, p. 472, reprinted from *Elbinger Anzeiger*, no. 85, 18 Oct. 1854.

Until lately no one but Shanks seems to have published any original computed value of π beyond 500D. In R. So. London, *Proc.*, v. 21, 1873, p. 319, Shanks gave a value to 707D but this was correct to only 459D; there were errors also in three following places. In the next volume of the *Proc.*, 22, p. 45, Shanks gave an amended value to 707D in which there was an error in the 326th decimal place, followed by an unexplained change of a 7 into a 1, in the 680th decimal place. The 707D have appeared in print correctly (so far as Shanks found them) in at least seven places:

(i) *Z. f. math. u. naturw. Unterricht*, v. 26, 1895, p. 263;

(ii) *L'Intermédiaire d. Mathématiciens*, v. 2, 1895, p. 389;

(iii–vi) G. PEANO, *Formulaire de Mathématique*, Turin, v. 2, no. 3, 1899, p. 143; v. 3, 1901, p. 176; v. 4, *Formulaire Mathématique*, 1902, p. 234; also v. 5, *Formulario Mathematico*, 1908, p. 256;

(vii) J. T. PETERS, *Zehnstellige Logarithmentafel*, v. 1, Berlin, 1922, *Anhang*, p. 1.

For the first time, so far as we know, the Shanks' computation of π beyond the 500th decimal place has recently been checked, with interesting results, by Mr. D. F. FERGUSON, of the Royal Naval College, Eaton, Chester,

England. In *Nature*, v. 157, 16 Mar. 1946, p. 342, he tells us that he used a formula, discovered by his colleague R. W. MORRIS,

$$\frac{\pi}{4} = 3\tan^{-1}\frac{1}{4} + \tan^{-1}\frac{1}{20} + \tan^{-1}\frac{1}{1985}, \tag{5}$$

and found that Shanks' value was incorrect, beginning with the 528th decimal place. He then gives both values from the 521st to the 540th decimal places. Mr. Ferguson has kindly supplied us with his calculated values from the 526th through the 620th decimal place, as follows:

S: 39501 60924 48077 23094 36285 53096 62027 55693 97986 95022
F: 39494 63952 24737 19070 21798 60943 70277 05392 17176 29317

S: 24749 96206 07497 03041 23668 86199 51100 89202 38377...
F: 67523 84674 81846 76694 05132 00056 81271 45263 56082...

J. W. WRENCH, JR. informed us that the relation (5) is not new, by any means; that it was given, for example, by S. L. LONEY, in his *Plane Trigonometry*, Cambridge, 1893, p. 277, and again by CARL STØRMER in *Archiv f. Mathem. og Naturv.*, v. 19, no. 3, 1896, p. 70, of his monograph to which we have already referred (*MTAC*, v. 2, p. 28), "Sur l'application de la théorie des nombres entiers complexes à la solution en nombres rationnels $x_1x_2 \cdots x_nc_1c_2 \cdots c_n$, k de l'équation $c_1 \operatorname{arctg} x_1 + c_2 \operatorname{arctg} x_2 + \cdots + c_n \operatorname{arctg} x_n = k\pi/4$." Here are tables of 102 three-term (left-hand member) equations, $k \neq 0$.

R. C. A.

57. GARRARD'S TABLES.—Brown University has recently acquired a copy of WILLIAM GARRARD, *Copious Trigonometrical Tables showing the results in all cases of Plane Trigonometry, By Inspection. Intended to complete the Requisite Tables to the Nautical Almanack; and as a Necessary Companion to the Theodolite*, London, 1789, 303 p. 15.3 × 24.4 cm. Garrard states that he was "late assistant observer at the Royal Observatory, Greenwich." Table I, the largest (225 p.), a navigation Traverse Table, for every hypotenuse, d (distance) = 1(1)300, of a plane right triangle are given, to 2D, the value of the base (L = latitude) and altitude (D = departure) for each angle 0(10′)90°. Table II (22 p.) is a similar table for every angle $0(\frac{1}{4})7\frac{3}{4}$ points. Table III (26 p.), for d = 1(1)10 are given to 5D, the value of L and D, for 0(10′)90°. T. IV (10 p.) is "an improved table of Meridional Parts for more accurately solving the cases of Mercator's Sailing." Then there are 3 p. of "Errata to the Traverse Tables in the last edition of Robertson's *Navigation*." This reference is doubtless to the enlarged fourth edition of JOHN ROBERTSON, *Elements of Navigation*, "carefully revised and corrected by WILLIAM WALES, Master of the Royal Mathematical School, Christ's Hospital, London." The traverse tables in v. 2, London, 1780, p. 81–130, cover the material of Garrard's T. I–II, for d = 1(1)120, and are possibly due to William Wales (1734?–1798), fellow of the Royal Society, which sent him in 1769 to observe at Hudson Bay the transit of Venus, and author of various works including a restoration of the Determinate Section of Apollonius of Perga (London, 1772). For further information about

Wales, see C. HUTTON, *Phil. and Math. Dict.*, second ed. 1815; E. I. CARLYLE, *Dict. Nat. Biog.*, v. 59, 1899; *Math. Gazette*, v. 14, 1929, p. 388; and *Mechanics' Mag.*, v. 60, 1854, p. 436–437. Garrard's T. IV of Meridional Parts is for each minute of the quadrant to 2D; Robertson's table, p. 215–224, for the same range, is to 1D. John Robertson (1712–1776) became a fellow of the Royal Society in 1741, and was its Librarian for the last few years of his life.

R. C. A.

58. TABLES OF POWERS N^p. Among many tables of this type, p non-integral, attention may be directed to one which seems to be comparatively unknown to mathematicians. It was calculated by L. A. BARRY and published in JOHN GOODMAN, *Mechanics Applied to Engineering*, London, New York & Toronto, Longmans, v. 2, 1927 [new impression 1941; reprint 1943], p. 450–469, 12.0 × 18.4 cm. The table gives to 4S, without differences, values of N^p where $N = 1.1(.1)30$ and $p = 1.1(.1)4$. As a one-volume work, Goodman's book goes back to the last century; but the second volume, which consists chiefly of worked examples, was first published in 1927.

Another, though smaller, table in which p takes a range of fractional values is that given in JAHNKE & EMDE, *Tables of Functions* (1933 edition and Dover reprints), and repeated in F. EMDE, *Tables of Elementary Functions*, 1940, p. 8 (see *MTAC*, v. 1, p. 384). This gives N^p to 3–4D, without differences, for $N = .1, .5(.1)1(.2)3(.5)5, 10$ and $p = .05(.05)1$. On errors in this table, see *MTAC*, v. 2, p. 47.

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QUERIES

18. ADDITIONS TO THE RUDOLPHINE TABLES.—Kepler's last great work, *Tabulæ Rudolphinæ, Quibus Astronomicæ Scientiæ, Temporum longinquitate collapsae Restauratio continetur* was published at Ulm in 1627, and there were later editions including one in English. To this original work three additions were later made, namely: (a) in 1629, 8 pages (p. 121–128), by Kepler, of "Sportula Genethliacis missa de Tabularum Rudolphi usu in computationibus astrologicis: cum modo dirigendi novo et naturali"; (b) A celebrated "Mappa Mundi Universalis," dated, 1630, but mysteriously dedicated to Kaiser Leopold, who did not commence to reign until 1658; (c) An Appendix, 46 p., by Kepler's son-in-law, J. BARTSCH, published at Sagan in 1630. In Providence, R. I. there are three copies of Kepler's work all including (a), and two of them also including (b). In MAX CASPAR, *Bibliographia Kepleriana* (Munich, 1936), where libraries having Kepler's works are listed, are the names of 9 German libraries owning copies of Kepler's work with (c). Where are copies with (c) in other countries? Tables in this Appendix, as well as in the Rudolphine Tables, are of value as exhibiting the spread of Napier's ideas on the continent, and as the basis of the final edition of a volume containing material due to Kepler and Bartsch, and published at Strassburg in 1700.

R. C. A.

QUERIES—REPLIES

24. TABLES OF $\tan^{-1}(m/n)$ (Q14, v. 1, p. 431; QR18, v. 1, p. 460; QR20, v. 2, p. 62).—In an earlier note by one of us (QR20) methods have been indicated by which the formula

$$\tan^{-1}(p+\lambda) - \tan^{-1} p = \tan^{-1}\frac{\lambda}{1+\lambda p+p^2} = \tfrac{1}{2}\pi - \tan^{-1}\frac{p^2+\lambda p+1}{\lambda}$$

may be used to make a table of values of $\tan^{-1}(m/n)$.

Another systematic use of this formula is suggested by the series

$$\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{7} + \tan^{-1}\frac{1}{13} + \cdots + \tan^{-1}\frac{1}{n^2+n+1}$$
$$= \tan^{-1}(n+1) - \tan^{-1} 1 = \tan^{-1}\frac{n}{n+2},$$

given, for instance, as an exercise in S. L. Loney's *Plane Trigonometry*, Part 2, 1900, Cambridge University Press, p. 126, or his *Key to Plane Trigonometry*, Part 2, 1912, C. U. Press, p. 91. This is proved by noting that

$$\tan^{-1}(n+1) - \tan^{-1} n = \tan^{-1}\frac{1}{n^2+n+1}.$$

For our purpose this may be generalized and rewritten as

$$\tan^{-1}\left(1+\frac{p}{n}\right) - \tan^{-1}\left(1+\frac{p}{n+1}\right)$$
$$= \tfrac{1}{2}\pi - \tan^{-1}\left(\frac{2n^2}{p} + 2\frac{p+1}{p}n + p + 1\right).$$

Then, with $p = 4$, for example, and $n = 1, 2, 3, \cdots$ we find

$$\tan^{-1} 5 - \tan^{-1} 3 = \tfrac{1}{2}\pi - \tan^{-1} 8$$
$$\tan^{-1} 3 - \tan^{-1} 7/3 = \tfrac{1}{2}\pi - \tan^{-1} 12$$
$$\tan^{-1} 7/3 - \tan^{-1} 2 = \tfrac{1}{2}\pi - \tan^{-1} 17$$
$$\tan^{-1} 2 - \tan^{-1} 9/5 = \tfrac{1}{2}\pi - \tan^{-1} 23$$
$$\tan^{-1} 9/5 - \tan^{-1} 5/3 = \tfrac{1}{2}\pi - \tan^{-1} 30$$

and so on. These may be used in succession (equivalent to summing the first few) as a check on the values occurring on the left of the equations; all the values on the right are tabulated, for instance, in the NYMTP *Table of Arc Tan x*. This check is useful because it progresses across columns of values of $\tan^{-1}(m/n)$ for constant n (the method in the earlier note was primarily for going down columns with constant n). The relations can also be used for actual evaluation of entries in the final table, but in this case, to avoid accumulation of errors, they would not be used in succession from the beginning, but backwards or forwards from the nearest value on the left that is given directly in the tables. Thus $\tan^{-1} 7/3$ could be found from $\tan^{-1} 3$ and $\tan^{-1} 12$, or from $\tan^{-1} 2$ and $\tan^{-1} 17$, both values used being taken from the NYMTP table.

With $p = 6$ and $n = 1, 2, 3, \cdots$ we find

$$\tan^{-1} 7 - \tan^{-1} 4 = \tfrac{1}{2}\pi - \tan^{-1} 29/3$$
$$\tan^{-1} 4 - \tan^{-1} 3 = \tfrac{1}{2}\pi - \tan^{-1} 13$$
$$\tan^{-1} 3 - \tan^{-1} 5/2 = \tfrac{1}{2}\pi - \tan^{-1} 17$$
$$\tan^{-1} 5/2 - \tan^{-1} 11/5 = \tfrac{1}{2}\pi - \tan^{-1} 65/3$$
$$\tan^{-1} 11/5 - \tan^{-1} 2 = \tfrac{1}{2}\pi - \tan^{-1} 27$$

and the relations cannot be used in succession for checking in the same way as for $n = 4$, since some of the values on the right are missing from the tables. It is still possible to work backwards and forwards from NYMTP tabular values as with $n = 4$.

Most early values of p give useful sets of formulae, but for $p = 7, 9, 11, 13, 14, 17$ ... the relations are not of much practical use. Similarly other series may be generalized, but it does not seem useful to give more details.

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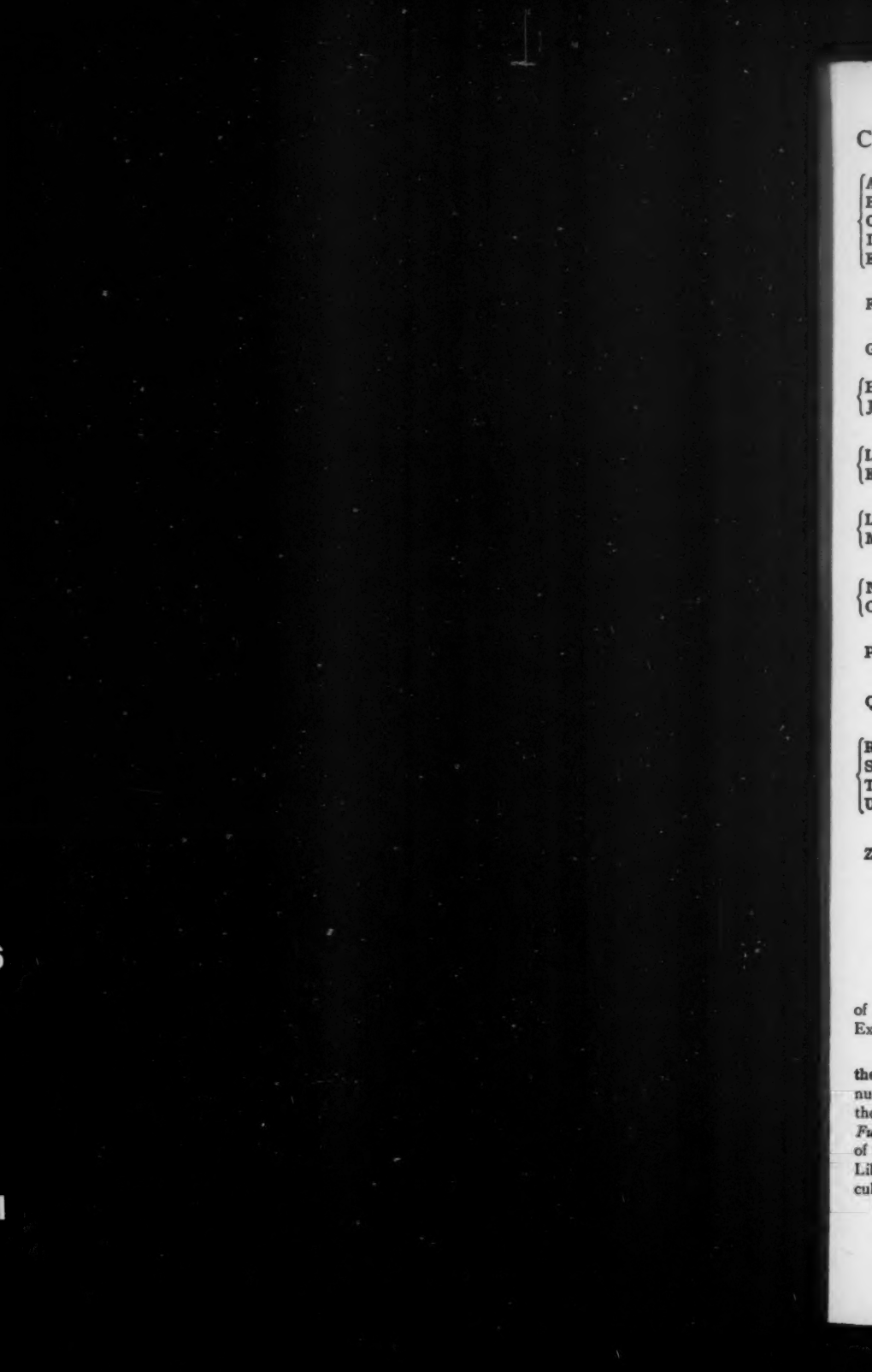
CORRIGENDA

P. 215, IB_6 7, *for* $x = .1$, *read* $x = 0, 1$; in line 2 *add* also $2y + \frac{1}{2}x^2 - \frac{1}{4} - \sum$, to 6D.
P. 255, B_9 13 *for* $2^{\frac{1}{2}}$, *read* $2^{-\frac{1}{2}}$.
P. 256 Transfer item C3 to be before the present first entry in D_1; eliminate and Airey from the heading of C.
P. 257, E 13, *for* 100, *read* 100^{ν}.
P. 280, under Backhaus 1, *for* v. 19, *read* v. 17.
P. 288, under Euler, *for* 1769, *read* 1748.
P. 301, under Ollendorff & Seeliger, *for* 518, *read* 578.
P. 303, under Roder, *for* [III], *read* [I]; and *for* 10, *read* 19; and add entries in I A_1, and C_1 for $J_n(x)$, $n = 0(1)4$, $x = [0(.1)1(.2)2.8, 5.6; 4D]$.
V. 2, p. 71, l. 18, *for* than ν, *read* than ξ.
P. 85, l. –6, *for* L. J. Cunningham, *read* L. E. Cunningham.



CLASSIFICATION OF TABLES, AND SUBCOMMITTEES

A. Arithmetical Tables. Mathematical Constants
B. Powers
C. Logarithms
D. Circular Functions
E. Hyperbolic and Exponential Functions
Professor DAVIS, *chairman*, Professor ELDER, Professor KETCHUM, Doctor LOWAN

F. Theory of Numbers
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G. Higher Algebra
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H. Numerical Solution of Equations
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I. Finite Differences. Interpolation
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